MAI4CAREU

Master programmes in Artificial Intelligence 4 Careers in Europe

Human Reasoning and the Weak Completion Semantics

TECHNISCHE UNIVERSITÄT DRESDEN

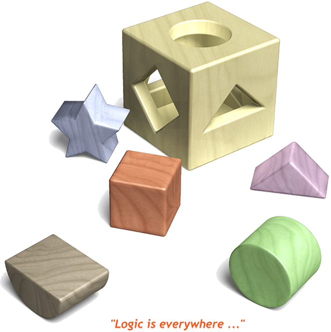
Co-financed by the European Union
Connecting Europe Facility

This Master is run under the context of Action No 2020-EU-IA-0087, co-financed by the EU CEF Telecom under GA nr. INEA/CEF/ICT/A2020/2267423

Foundations – Logics

Steffen Hölldobler
Technische Universität Dresden, Germany
North Caucasus Federal University, Russian Federation

- Alphabet
- Formulas
- Equations
- Interpretations
- Models
- Semantic Equivalence
- Logical Consequence



"Logic is everywhere ..."

Alphabet

- We consider an alphabet consisting of
  - a finite set of function symbols with arity ≥ 0
  - a countably infinite set of variables
  - a finite or countably infinite set of relation symbols with arity ≥ 0
  - the connectives $\neg$, $\wedge$, $\vee$, $\leftarrow$, and $\leftrightarrow$
  - the existential quantifier $\exists$
  - the universal quantifier $\forall$
  - the special symbols (,), $\top$, $\bot$, U, and ,
- We assume that alphabets are implicitly given as the set of symbols occurring in the syntactic objects under consideration

Terms, Atoms, and Literals

- The set of terms is the smallest set satisfying the following conditions:
  - Each variable is a term
  - If f is an n-ary function symbol, $n \geq 0$, and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a term as well
    - $X,\ Y,\ a,\ b,\ f(a, X),\ g\,a$
- The set of atoms consists of all expressions of the form $p(t_1, \ldots, t_n)$, where p is an n-ary relation symbol, $n \geq 0$, and $t_1, \ldots, t_n$ are terms
  - $p(b, X),\ q,\ r,\ s\,a$
- A literal is either an atom or its negation
  - $p(a, b),\ \neg p(a, b)$
- A term, atom, or literal is said to be ground iff if it does not contain the occurrence of a variable

Formulas

- The set of formulas is the smallest set satisfying the following conditions
  - Each atom and each truth constant is a formula
  - If F is a formula, then so is $\neg F$
  - If F and G are formulas, then so are $(F \wedge G)$, $(F \vee G)$, $(F \leftarrow G)$, and $(F \leftrightarrow G)$
  - If F is a formula and X is a variable, then $(\forall X)\ F$ and $(\exists X)\ F$ are formulas
- A formula is called clause iff if it is a finite disjunction of literals
- In logic programming, clauses are often written in the form

$$A \leftarrow L_1 \wedge \ldots \wedge L_n$$

where A is an atom and L_i, $1 \leq i \leq n$, are literals

  - They are called program clauses with head A and body $L_1 \wedge \ldots \wedge L_n$

Equations

- An equation is an atom of the form $s \approx t$, where
  - s and t are terms and
  - $\approx$ is a binary relation symbol written infix
- Equations are assumed to be universally closed
- We usually consider sets of equations
- Examples
  - $\{a \approx b\}$
  - $\{X \circ 1 \approx X,\ X \circ Y \approx Y \circ X,\ (X \circ Y) \circ Z \approx X \circ (Y \circ Z)\}$ **AC1**

Axioms of Equality

- The equality relation enjoys some typical properties
- They are specified in the following logic program

$$
\begin{array}{rcll}
X \approx X & \leftarrow & \top & \text{reflexivity} \\
X \approx Y & \leftarrow & Y \approx X & \text{symmetry} \\
X \approx Z & \leftarrow & X \approx Y \wedge Y \approx Z & \text{transitivitiy} \\
f(X_1, \ldots, X_n) \approx f(Y_1, \ldots, Y_n) & \leftarrow & \bigwedge_{i=1}^{n} X_i \approx Y_i & \text{f-substitutivity} \\
r(Y_1, \ldots, Y_n) & \leftarrow & r(X_1, \ldots, X_n) \wedge \bigwedge_{i=1}^{n} X_i \approx Y_i & \text{r-substitutivity}
\end{array}
$$

where substitutivity axioms are given for each function and relation symbol

Equational Theories

- An **equational theory** consists of a set ot equations and the axioms of equality
- It is specified by the set of equations
- **Example**

$$\{X \circ 1 \approx X,\ X \circ Y \approx Y \circ X,\ (X \circ Y) \circ Z \approx X \circ (Y \circ Z)\}$$

specifies the **AC1 theory**

Finest Congruence Relation

- An equational theory defines a finest congruence relation $\cong$ on the set of ground terms
  - An equational theory is a definite logic program
  - Definite logic programs enjoy the model intersection property
  - The least model is the finest congruence relation
- Let t be a ground term
  - $[t]$ denotes the congruence class defined by $\cong$ and containing t
  - If the set of equations is empty we write t instead of $[t]$

An Abbreviation

- $[p(t_1, \ldots, t_n)]$ is an abbreviation for $p([t_1], \ldots, [t_n])$
- $[p(t_1, \ldots, t_n)] = [q(s_1, \ldots, s_m)]$ iff
  - $p = q$
  - $n = m$ and
  - $[t_i] = [s_i]$ for all $1 \leq i \leq n$
- If the set of equations is empty we write $p(t_1, \ldots, t_n)$ instead of $[p(t_1, \ldots, t_n)]$
- **Example** Consider the AC1-theory

$$\begin{array}{rcl} [d \circ t_2] & = & [t_2 \circ d] \\ [d \circ t_1 \circ d] & = & [t_1 \circ d \circ d \circ 1] \\ [p(d \circ t_2, d \circ t_1 \circ d)] & = & [p(t_2 \circ d, t_1 \circ d \circ d \circ 1)] \end{array}$$

where d, t_2, t_1, 1 are constants, $\circ$ is a function, and p is a relation symbol

Interpretations and Models

- The Herbrand universe is the quotient of the set of ground terms modulo $\cong$
- The Herbrand base is the of all expressions of the form $[p(t_1, \ldots, t_n)]$ where
  - p is an n-ary relation symbol and
  - $[t_i]$ are elements of the Herbrand universe for all $1 \leq i \leq n$
- An interpretation is a mapping from the set of formulas into the set of truth values such that
  - truth constants are mapped onto themselves and
  - a given equational theory is mapped to true
- An interpretation is defined by
  - the truth tables for the connectives and
  - the mapping of the Herbrand base to the truth values
- An interpretation I is a model for a formula F ($I \models F$) iff I maps F to true

Interpretations and Models – Example

- Consider $\mathcal{P} = \{q\,X \leftarrow \neg\, p\,X,\ p\,a \leftarrow \top\}$ and $\mathcal{E} = \{a \approx b\}$
- The Herbrand universe is $\{[a]\}$
- The Herbrand base is $\{[p\,a],\ [q\,b]\}$
- Interpretations are given by the truth tables for the connectives and
  - $[p\,a] \mapsto \top \quad [q\,b] \mapsto \top$
  - $[p\,a] \mapsto \bot \quad [q\,b] \mapsto \top$
  - $[p\,a] \mapsto \top \quad [q\,b] \mapsto \bot$
  - $[p\,a] \mapsto \bot \quad [q\,b] \mapsto \bot$
- Which interpretations are models for $\mathcal{P}$ and $\mathcal{E}$ in classical two-valued logic?

Semantic Equivalence

- **Two formulas F and G are semantically equivalent ($F \equiv G$)**
  iff for all interpretations I we find $I\,F = I\,G$

  - **Under which logics is $\bot \vee F \equiv F$ and $(F \leftarrow G) \wedge (G \leftarrow F) \equiv (F \leftrightarrow G)$?**
    - **Two-valued classical logic**
    - **Three-valued Łukasiewicz logic**
    - **Three-valued Kleene logics**
    - **Three-valued Fitting logic**

  Prove your claim

Logical Consequence

- Let $\mathcal{F}$ be a set of formulas and G a formula
- $\mathcal{F}$ logically entails G or G is a logical consequence of $\mathcal{F}$ ($\mathcal{F} \models G$)
  iff every model for $\mathcal{F}$ is also a model for G
- Consider two-valued classical logic
  - Does $\{\ell \leftarrow e,\ e\} \models \ell$ hold?
  - Does $\{\ell \leftarrow e,\ \neg e\} \models \neg \ell$ hold?
  - Does $\{\ell \leftarrow e,\ \ell\} \models e$ hold?
  - Does $\{\ell \leftarrow e,\ \neg \ell\} \models \neg e$ hold?

  Prove your claim

MAI4CAREU

Master programmes in Artificial Intelligence 4 Careers in Europe

Co-financed by the European Union
Connecting Europe Facility

This Master is run under the context of Action No 2020-EU-IA-0087, co-financed by the EU CEF Telecom under GA nr. INEA/CEF/ICT/A2020/2267423