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Master programmes in Artificial Intelligence 4 Careers in Europe

Human Reasoning and the Weak Completion Semantics

TECHNISCHE UNIVERSITÄT DRESDEN

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Foundations – Fixed Point Theory

Steffen Hölldobler
Technische Universität Dresden, Germany
North Caucasus Federal University, Russian Federation

- **Relations and Partial Orders**
- **Complete Partial Orders**
- **Monotonic and Continuous Mappings**
- **Fixed Point Theorems**
- **Ordinal Numbers**
- **Finite Complete Partial Orders**

"Logic is everywhere ..."

Relations and Partial Orders

- Let $\mathcal{S}$ be a set
- A (binary) relation R on $\mathcal{S}$ is a subset of $\mathcal{S} \times \mathcal{S}$
  - xRy denotes $(x, y) \in R$
- A relation R on $\mathcal{S}$ is a partial order if the following conditions hold
  - Reflexivity for all $x \in \mathcal{S}$ we find xRx
  - Antisymmetry for all $x, y \in \mathcal{S}$ we find $x = y$ if xRy and yRx
  - Transitivity for all $x, y, z \in \mathcal{S}$ we find xRz if xRy and yRz

  $\mathcal{S}$ is said to be a partially ordered set wrt the partial order R
- Examples
  - $\mathbb{N}$ is a partially ordered set wrt $\leq$
  - $2^{\{e, \ell, ab_e\}}$ is a partially ordered set wrt $\subseteq$
- In the sequel, let $\leq$ denote a partial order

Upper and Lower Bounds

- Let $\mathcal{S}$ be a partially ordered set wrt $\leq$, $a, b \in \mathcal{S}$, and $\mathcal{X} \subseteq \mathcal{S}$

- a is an **upper bound** of $\mathcal{X}$ if for every $x \in \mathcal{X}$ we have $x \leq a$
- a is the **least upper bound** of $\mathcal{X}$ if
  - a is an upper bound of $\mathcal{X}$ and
  - for every upper bound a' of $\mathcal{X}$ we have $a \leq a'$
- *lub* $\mathcal{X}$ denotes the least upper bound of $\mathcal{X}$ if it exists

- b is a **lower bound** of $\mathcal{X}$ if for every $x \in \mathcal{X}$ we have $b \leq x$
- b is the **greatest lower bound** of $\mathcal{X}$ if
  - b is a lower bound of $\mathcal{X}$ and
  - for every lower bound b' of $\mathcal{X}$ we have $b' \leq b$
- *glb* $\mathcal{X}$ denotes the greatest lower bound of $\mathcal{X}$ if it exists

Directed Sets

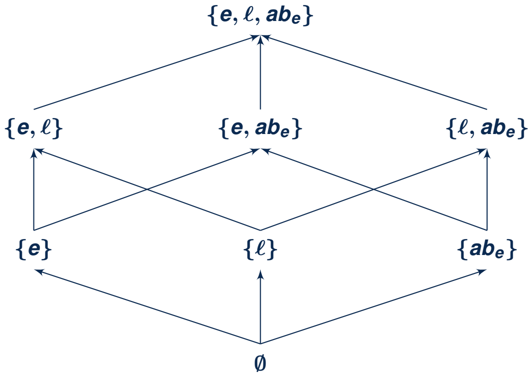
- Let $\mathcal{S}$ be a partially ordered set wrt $\leq$ and $\mathcal{X}$ a non-empty subset of $\mathcal{S}$
- $\mathcal{X}$ is **directed**
  if for every $x, y \in \mathcal{X}$ there exists some $z \in \mathcal{X}$ such that $x \leq z$ and $y \leq z$
- **Example** Let $\mathcal{A} = \{e, \ell, ab_e\}$ and consider $2^{\mathcal{A}}$ wrt $\subseteq$
  - Let $\mathcal{X}_1 = \{\emptyset, \{e\}, \{\ell\}\}$
  - Let $\mathcal{X}_2 = \{\emptyset, \{e\}, \{\ell\}, \{e, \ell\}\}$
  - Do least upper and greatest lower bounds exist for these sets?
  - Are these sets directed?

Complete Partial Orders

- A partially ordered set $\mathcal{S}$ wrt $\leq$ is a complete partial order if
  - $\mathcal{S}$ has a least element and
  - for every directed subset $\mathcal{X}$ of $\mathcal{S}$ there exists *lub* $\mathcal{X} \in \mathcal{S}$
- Is $2^{\{e,\ell,ab_e\}}$ wrt $\subseteq$ a complete partial order?

Complete Partial Orders – Hasse Diagram

- **Consider $2^{\{e,\ell,ab_e\}}$ wrt $\subseteq$**



Monotonic and Continuous Mappings

- Let $\mathcal{S}$ be a partially ordered set wrt $\leq$ and $f : \mathcal{S} \rightarrow \mathcal{S}$ a mapping
- f is monotonic if for every $x, y \in \mathcal{S}$ such that $x \leq y$ we have $f\,x \leq f\,y$
- f is continuous if for every directed subset $\mathcal{X}$ of $\mathcal{S}$ we find

$$f\;lub\;\mathcal{X} = lub\;\{f\,z \mid z \in \mathcal{X}\}$$

- **Proposition 1** Let $\mathcal{S}$ be a partially ordered set
  Every continuous mapping $f : \mathcal{S} \rightarrow \mathcal{S}$ is monotonic
- **Proof** Consider continuous $f : \mathcal{S} \rightarrow \mathcal{S}$ and some $x, y \in \mathcal{S}$ such that $x \leq y$
  - $\mathcal{X} = \{x, y\}$ is a directed subset of $\mathcal{S}$ and $lub\;\mathcal{X} = y$
  - By the continuity of f we learn

$$f\,y = f\;lub\;\{x, y\} = f\;lub\;\mathcal{X} = lub\;\{f\,z \mid z \in \mathcal{X}\} = lub\;\{f\,x, f\,y\}$$

  - Consequently $f\,x \leq f\,y$ which shows that f is monotonic □

Fixed Points and the Knaster-Tarski Fixed Point Theorem

- $x \in \mathcal{S}$ is a fixed point of f iff $f\,x = x$
- $x \in \mathcal{S}$ is the least fixed point of f iff
  - x is a fixed point of f and
  - $x \leq y$ for all fixed points $y \in \mathcal{S}$ of f
- Theorem 2 (Knaster-Tarski)
  Let $\mathcal{S}$ be a complete partial order and f a monotonic mapping on $\mathcal{S}$
  Then f has a least fixed point
- How can the least fixed point be computed?

An Informal Introduction to Ordinal Numbers

- **Ordinal numbers are a generalization of the natural numbers**

$$0,\ 1,\ 2,\ 3,\ \ldots$$

are non-limit ordinal numbers

$$\omega$$

is the first limit ordinal number

$$\omega + 1,\ \omega + 2,\ \omega + 3,\ \ldots$$

are the next non-limit ordinal numbers

$$\omega + \omega$$

is the second limit ordinal number, and so on

Iterating Functions on Complete Partial Orders

- Let S be a complete partial order with least element $\diamond$ and f a mapping on S
- We define

$$\begin{array}{lcll} f \uparrow 0 & = & \diamond, & \\ f \uparrow \alpha & = & f(f \uparrow (\alpha - 1)) & \text{if } \alpha \text{ is a non-limit ordinal and } \alpha \neq 0, \\ f \uparrow \alpha & = & lub\{f \uparrow \beta \mid \beta < \alpha\} & \text{if } \alpha \text{ is a limit ordinal} \end{array}$$

- **Proposition 3** Let $\mathcal{S}$ be a complete partial order, f a monotonic mapping on $\mathcal{S}$ and x the least fixed point of f. Then, for some ordinal γ we find $x = f \uparrow \gamma$
- **Can you construct an example where we need to iterate beyond the first limit ordinal to obtain the least fixed point?**

Kleene Fixed Point Theorem

- **Theorem 4 (Kleene)**
  Let $\mathcal{S}$ be a complete partial order and f a continuous mapping on $\mathcal{S}$
  Then, $f \uparrow \omega$ is the least fixed point of f
- **Can you construct an example where the least fixed point is reached in**
  - **less than ω steps?**
  - **precisely ω steps?**

Finite Complete Partial Orders

- **Lemma 5** Let $\mathcal{X}$ be a directed set and $\mathcal{Y}$ be a finite subset of $\mathcal{X}$
  Then, $\mathcal{X}$ contains an upper bound of $\mathcal{Y}$
- **Proof** $\leadsto$ **Exercise**
- **Corollary 6** Any finite directed set contains its own least upper bound
- **Proposition 7**
  Let $\mathcal{S}$ be a finite complete partial order and f a monotonic mapping on $\mathcal{S}$
  Then f is continuous
- **Proof** $\leadsto$ **Exercise**
- **Corollary 8**
  Let $\mathcal{S}$ be a finite complete partial order and f be a monotonic mapping on $\mathcal{S}$
  Then $f \uparrow \omega$ is the least fixed point of f

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