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Human Reasoning and the Weak Completion Semantics

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The Weak Completion Semantics – Theory

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- Programs
- Weakly Completed Programs
- The Meaning of Programs
- Computing Least Models
- Semantic Operators as Contraction Mappings
- Abduction

Programs

- A (normal logic) program $\mathcal{P}$ is a finite or countably infinite set of clauses of the form
$$A \leftarrow Body$$
  - A is an atom (but not an equation) called head
  - $Body$ is either a non-empty conjunction of literals, or $\top$, or $\bot$
- Clauses are assumed to be universally closed
- $A \leftarrow \top$ is called (positive) fact
- $A \leftarrow \bot$ is called (negative) assumption
- All other clauses are called rules
- $\mathcal{P}$ is propositional iff all atoms occurring in $\mathcal{P}$ are propositional
- $\mathcal{P}$ is a datalog program iff the terms occurring in $\mathcal{P}$ are variables and constants
- $\mathcal{P}$ is a definite program iff it does not contain an occurrence of $\bot$ or $\neg$

Programs – Example

- Let $\mathcal{P}$ be

$$
\begin{array}{rcl}
\ell & \leftarrow & e \wedge \neg ab_e \\
\ell & \leftarrow & t \wedge \neg ab_t \\
e & \leftarrow & \top \\
ab_e & \leftarrow & \bot \\
ab_t & \leftarrow & \bot
\end{array}
$$

Alphabet

- Let $\mathcal{P}$ be a program and $\mathcal{E}$ an equational theory
  - The alphabet consists precisely of the symbols occurring in $\mathcal{P}$ and $\mathcal{E}$
  - If $\mathcal{P}$ or $\mathcal{E}$ is a first-order program
    then the alphabet must contain at least one constant symbol

Ground Instances

- A **ground instance of a clause** is obtained by replacing each variable occurring in the clause by a ground term
  - The replacement must be consistent in that multiple occurrences of the same variable are replaced by the same ground term
- The **ground instance of a program** $\mathcal{P}$ is the set of all ground instances of clauses occurring in $\mathcal{P}$
  - $g\mathcal{P}$ denotes the ground instance of $\mathcal{P}$
  - If $\mathcal{P}$ is a propositional program then $g\mathcal{P} = \mathcal{P}$

Ground Programs – Example

- Let $\mathcal{P}$ be

$$\begin{array}{rcl} q\,a & \leftarrow & \top \\ q\,s\,X & \leftarrow & q\,X \end{array}$$

- Then $g\mathcal{P}$ consists of

$$\begin{array}{rcl} q\,a & \leftarrow & \top \\ q\,s\,a & \leftarrow & q\,a \\ q\,s\,s\,a & \leftarrow & q\,s\,a \\ \vdots & \vdots & \vdots \end{array}$$

- Is $q\,s\,a \leftarrow q\,s\,a \in g\mathcal{P}$?

Defined Ground Atoms

- Let $\mathcal{P}$ be a ground program, $\mathcal{E}$ an equational theory, and A a ground atom
  - If $\mathcal{E}$ is empty, then A is defined in $\mathcal{P}$ iff $\mathcal{P}$ contains a clause with head A
  - If $\mathcal{E}$ is not empty, then A is defined in $\mathcal{P}$ iff $\mathcal{P}$ contains a clause with head A' and $[A] = [A']$
- A is undefined in $\mathcal{P}$ iff A is not defined in $\mathcal{P}$
- *def* $\mathcal{P}$ denotes the set of defined atoms in $\mathcal{P}$

Defined Ground Atoms – Examples

- **Consider the following programs**

$$\mathcal{E} = \emptyset$$

$$\begin{array}{rcl} \ell & \leftarrow & e \wedge \neg ab_e \\ \ell & \leftarrow & t \wedge \neg ab_t \\ e & \leftarrow & \top \\ ab_e & \leftarrow & \bot \\ ab_t & \leftarrow & \bot \end{array}$$

$$\mathcal{E} = \{a \approx b\}$$

$$\begin{array}{lcl} p\,a & \leftarrow & \top \\ q\,c & \leftarrow & \bot \end{array}$$

  - How does $def\ \mathcal{P}$ look like?
  - Are there any undefined atoms?

Definitions

- Let $\mathcal{P}$ be a ground program, $\mathcal{E}$ an equational theory, and $\mathcal{S}$ a consistent set of literals
  - If $\mathcal{E}$ is empty, then $\mathit{defs}(\mathcal{P}, \mathcal{S}) = \{A \leftarrow \mathit{Body} \in \mathcal{P} \mid A \in \mathcal{S} \text{ or } \neg A \in \mathcal{S}\}$
  - If $\mathcal{E}$ is not empty,
    then $\mathit{defs}(\mathcal{P}, \mathcal{S}) = \{A' \leftarrow \mathit{Body} \in \mathcal{P} \mid A \in \mathcal{S} \text{ or } \neg A \in \mathcal{S} \text{ and } [A'] = [A]\}$
- Let $\mathcal{P}$ be

$$
\begin{array}{rcl}
\ell & \leftarrow & e \wedge \neg ab_e \\
\ell & \leftarrow & o \wedge \neg ab_o \\
e & \leftarrow & \top \\
ab_e & \leftarrow & \bot \\
ab_o & \leftarrow & \bot \\
ab_e & \leftarrow & \neg o \\
ab_o & \leftarrow & \neg e
\end{array}
$$

  - How does $\mathit{defs}(\mathcal{P}, \{e, \neg ab_e\})$ look like?

Assumptions

- Let $\mathcal{P}$ be a ground program, $\mathcal{E}$ an equational theory, and A a ground atom
  - If $\mathcal{E}$ is empty then $\neg A$ is assumed in $\mathcal{P}$ iff
    - $\mathcal{P}$ contains an assumption with head A and
    - $\mathcal{P}$ does neither contain a fact $A \leftarrow \top$ nor a rule $A \leftarrow Body$
  - If $\mathcal{E}$ is not empty then $\neg A$ is assumed in $\mathcal{P}$ iff
    - $\mathcal{P}$ contains an assumption of the form $A' \leftarrow \bot$ with $[A] = [A']$ and
    - $\mathcal{P}$ does neither contain a fact $B \leftarrow \top$ nor a rule $B \leftarrow Body$ with $[A] = [B]$
- Why has the second condition been added?

Assumptions – Examples

- What is assumed in the following programs if $\mathcal{E} = \emptyset$?
  - $$\begin{array}{rcl} \ell & \leftarrow & e \wedge \neg ab_e \\ \ell & \leftarrow & t \wedge \neg ab_t \\ e & \leftarrow & \top \\ ab_e & \leftarrow & \bot \\ ab_t & \leftarrow & \bot \end{array}$$
  - $$\begin{array}{rcl} \ell & \leftarrow & e \wedge \neg ab_e \\ \ell & \leftarrow & o \wedge \neg ab_o \\ e & \leftarrow & \top \\ ab_e & \leftarrow & \bot \\ ab_o & \leftarrow & \bot \\ ab_e & \leftarrow & \neg o \\ ab_o & \leftarrow & \neg e \end{array}$$
- **Assumptions can be overridden**

Weakly Completed Programs

- Let $\mathcal{P}$ be a ground program and $\mathcal{E}$ an equational theory
- Consider the following transformation

1 For all $A \in \mathit{def}\,\mathcal{P}$ do

- If $\mathcal{E}$ is empty, replace all clauses of the form $A \leftarrow Body_1,\ A \leftarrow Body_2,\ \ldots$ occurring in $\mathcal{P}$ by $A \leftarrow Body_1 \vee Body_2 \ldots$
- If $\mathcal{E}$ is not empty, replace all clauses of the form $A_1 \leftarrow Body_1,\ A_2 \leftarrow Body_2,\ \ldots$ occurring in $\mathcal{P}$ with $[A_1] = [A_2] = \ldots = [A]$ by $A \leftarrow Body_1 \vee Body_2 \ldots$

2 Add $A \leftarrow \bot$ for all undefined ground atoms A occurring in $\mathcal{P}$

3 Replace all occurrences of $\leftarrow$ by $\leftrightarrow$

- The resulting set is called completion of $\mathcal{P}$ or $c\,\mathcal{P}$
- If step 2 is omitted then the resulting set is called weak completion of $\mathcal{P}$ or $wc\,\mathcal{P}$

Program Completion – Example

- Let $\mathcal{P}$ be

$$\begin{array}{rcl} \ell & \leftarrow & e \wedge \neg ab_e \\ \ell & \leftarrow & t \wedge \neg ab_t \\ e & \leftarrow & \top \\ ab_e & \leftarrow & \bot \\ ab_t & \leftarrow & \bot \end{array}$$

- The weak completion of $\mathcal{P}$ consists of

$$\begin{array}{rcl} \ell & \leftrightarrow & (e \wedge \neg ab_e) \vee (t \wedge \neg ab_t) \\ e & \leftrightarrow & \top \\ ab_e & \leftrightarrow & \bot \\ ab_t & \leftrightarrow & \bot \end{array}$$

- The completion of $\mathcal{P}$ is obtained by adding

$$t \leftrightarrow \bot$$

Program Completion – Another Example

- Let $\mathcal{P}$ be

$$\begin{array}{lcl} p\,a & \leftarrow & \top \\ q\,b & \leftarrow & r\,b \end{array}$$

- How does $c\,\mathcal{P}$ look like?
- How does $wc\,\mathcal{P}$ look like?

Program Completion – Yet Another Example

- Let $\mathcal{P}$ be

$$
\begin{array}{rcl}
\ell & \leftarrow & e \wedge \neg ab_e \\
\ell & \leftarrow & o \wedge \neg ab_o \\
e & \leftarrow & \top \\
ab_e & \leftarrow & \bot \\
ab_o & \leftarrow & \bot \\
ab_e & \leftarrow & \neg o \\
ab_o & \leftarrow & \neg e
\end{array}
$$

- The weak completion of $\mathcal{P}$ consists of

$$
\begin{array}{rcl}
\ell & \leftrightarrow & (e \wedge \neg ab_e) \vee (o \wedge \neg ab_o) \\
e & \leftrightarrow & \top \\
ab_e & \leftrightarrow & \bot \vee \neg o \\
ab_o & \leftrightarrow & \bot \vee \neg e
\end{array}
$$

- Under Łukasiewicz logic we find $F \vee \bot \equiv F$

Convention

- Let $\mathcal{P}$ be a ground program and $\mathcal{E}$ an equational theory
- In the future
  - If $\mathcal{E}$ is empty, we will delete an assumption $A \leftarrow \bot$ if the program contains a fact $A \leftarrow \top$ or a rule $A \leftarrow \mathit{Body}$
  - If $\mathcal{E}$ is not empty, then $A \leftarrow \bot$ will be deleted if the ground program contains $B \leftarrow \top$ or $B \leftarrow \mathit{Body}$ with $[A] = [B]$

Sets of Literals versus Sets of Facts and Assumptions

- **Let $\mathcal{S}$ be a consistent set of ground literals**
- $\mathcal{S}^{\uparrow} = \{A \leftarrow \top \mid A \in \mathcal{S}\} \cup \{A \leftarrow \bot \mid \neg A \in \mathcal{S}\}$
- **Let $\mathcal{P}$ be a ground program containing only facts and assumptions**
  - **Remember our convention!**
- $\mathcal{P}^{\downarrow} = \{A \mid A \leftarrow \top \in \mathcal{P}\} \cup \{\neg A \mid A \leftarrow \bot \in \mathcal{P}\}$
- **Example** **Let $\mathcal{S} = \{e, \neg ab_e\}$ and $\mathcal{P} = \{e \leftarrow \top,\ ab_e \leftarrow \bot\}$**
  - $\mathcal{S}^{\uparrow} = \mathcal{P}$
  - $\mathcal{P}^{\downarrow} = \mathcal{S}$
- **Is $\mathcal{P}^{\downarrow}$ consistent?**

The Depends On Relation

- Let $\mathcal{P}$ be a ground program
- Atom A directly depends on atom B if
  - $\mathcal{P}$ contains a rule of the form $A \leftarrow Body$ and
  - B occurs (positively or negatively) in $Body$
- The depends on relation is the transitive closure of the directly depends on relation
- Example Let $\mathcal{P} = \{q\,a \leftarrow \top,\ q\,s\,a \leftarrow q\,a,\ q\,s\,s\,a \leftarrow q\,s\,a,\ \ldots\}$
  - $q\,s\,a$ directly depends on $q\,a$
  - $q\,s\,s\,a$ directly depends on $q\,s\,a$
  - $q\,s\,a$ depends on $q\,a$
  - $q\,s\,s\,a$ depends on $q\,s\,a$ and $q\,a$

The Function *deps*

- Let $\mathcal{P}$ be a ground program and $\mathcal{S}$ a consistent set of ground literals

$$deps(\mathcal{P}, \mathcal{S}) \quad = \quad \{B \leftarrow Body \in \mathcal{P} \mid Body \in \{\top, \bot\} \text{ and there exists } A \in \mathcal{S} \text{ or } \neg A \in \mathcal{S} \text{ such that } A \text{ depends on } B\}$$

- **Example** Let $\mathcal{P} = \{q\,a \leftarrow \top,\ q\,s\,a \leftarrow q\,a,\ q\,s\,s\,a \leftarrow q\,s\,a,\ \ldots\}$
  - $deps(\mathcal{P}, \{q\,s\,a\,a,\ \neg\, q\,s\,a\}) = \{q\,a \leftarrow \top\}$

The Meaning of Programs

- Let $\mathcal{P}$ be a program and $\mathcal{E}$ an equational theory
  - In many scenarios $\mathcal{E} = \emptyset$
  - When modeling ethical decision problems $\mathcal{E} = AC1$
- Recall equations, equational theories, interpretations, and models
- What is the meaning of $\mathcal{P}$?

Łukasiewicz Three-Valued Logic

F	$\neg F$
$\top$	$\bot$
$\bot$	$\top$
U	U

$\wedge$	$\top$	U	$\bot$
$\top$	$\top$	U	$\bot$
U	U	U	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$

$\vee$	$\top$	U	$\bot$
$\top$	$\top$	$\top$	$\top$
U	$\top$	U	U
$\bot$	$\top$	U	$\bot$

$\leftarrow$	$\top$	U	$\bot$
$\top$	$\top$	$\top$	$\top$
U	U	$\top$	$\top$
$\bot$	$\bot$	U	$\top$

$\leftrightarrow$	$\top$	U	$\bot$
$\top$	$\top$	U	$\bot$
U	U	$\top$	U
$\bot$	$\bot$	U	$\top$

Kleene Three-Valued Logic

F	$\neg F$
$\top$	$\bot$
$\bot$	$\top$
U	U

$\wedge$	$\top$	U	$\bot$
$\top$	$\top$	U	$\bot$
U	U	U	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$

$\vee$	$\top$	U	$\bot$
$\top$	$\top$	$\top$	$\top$
U	$\top$	U	U
$\bot$	$\top$	U	$\bot$

$\leftarrow$	$\top$	U	$\bot$
$\top$	$\top$	$\top$	$\top$
U	U	U	$\top$
$\bot$	$\bot$	U	$\top$

$\leftrightarrow$	$\top$	U	$\bot$
$\top$	$\top$	U	$\bot$
U	U	U	U
$\bot$	$\bot$	U	$\top$

Fitting Three-Valued Logic

F	$\neg F$
$\top$	$\bot$
$\bot$	$\top$
U	U

$\wedge$	$\top$	U	$\bot$
$\top$	$\top$	U	$\bot$
U	U	U	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$

$\vee$	$\top$	U	$\bot$
$\top$	$\top$	$\top$	$\top$
U	$\top$	U	U
$\bot$	$\top$	U	$\bot$

$\leftarrow$	$\top$	U	$\bot$
$\top$	$\top$	$\top$	$\top$
U	U	U	$\top$
$\bot$	$\bot$	U	$\top$

$\leftrightarrow$	$\top$	U	$\bot$
$\top$	$\top$	$\bot$	$\bot$
U	$\bot$	$\top$	$\bot$
$\bot$	$\bot$	$\bot$	$\top$

Three-Valued Interpretations

- A **(three-valued) interpretation** assigns to each formula a value from $\{\top, \bot, \mathsf{U}\}$
- It is represented by $\langle I^\top, I^\bot \rangle$, where
  - $I^\top$ contains all ground atoms which are mapped to $\top$
  - $I^\bot$ contains all ground atoms which are mapped to $\bot$
  - $I^\top \cap I^\bot = \emptyset$
  - All ground atoms which occur neither in $I^\top$ nor $I^\bot$ are mapped to U
- In the sequel, I, J denote interpretations $\langle I^\top, I^\bot \rangle$, $\langle J^\top, J^\bot \rangle$, respectively
- The **intersection** $I \cap J$ is defined as $\langle I^\top \cap J^\top, I^\bot \cap J^\bot \rangle$

Three-Valued Interpretations – Examples

- **Consider**

$\mathcal{P}$			$wc\,\mathcal{P}$			$c\,\mathcal{P}$		
ℓ	$\leftarrow$	$e \wedge \neg ab_e$	ℓ	$\leftrightarrow$	$(e \wedge \neg ab_e)$	ℓ	$\leftrightarrow$	$(e \wedge \neg ab_e)$
ℓ	$\leftarrow$	$t \wedge \neg ab_t$			$\vee\ (t \wedge \neg ab_t)$			$\vee\ (t \wedge \neg ab_t)$
e	$\leftarrow$	$\top$	e	$\leftrightarrow$	$\top$	e	$\leftrightarrow$	$\top$
ab_e	$\leftarrow$	$\bot$	ab_e	$\leftrightarrow$	$\bot$	ab_e	$\leftrightarrow$	$\bot$
ab_t	$\leftarrow$	$\bot$	ab_t	$\leftrightarrow$	$\bot$	ab_t	$\leftrightarrow$	$\bot$
						t	$\leftrightarrow$	$\bot$

- **Then**

I	$I\,\mathcal{P}$	$I\,wc\,\mathcal{P}$	$I\,c\,\mathcal{P}$
$\langle\{e, ab_e\}, \emptyset\rangle$	$\top$	$\bot$	$\bot$
$\langle\{e, \ell\}, \{ab_e, ab_t\}\rangle$	$\top$	$\top$	U
$\langle\{e, \ell, t\}, \{ab_e, ab_t\}\rangle$	$\top$	$\top$	$\bot$
$\langle\{e, \ell\}, \{ab_e, ab_t, t\}\rangle$	$\top$	$\top$	$\top$

Models

- An interpretation I is a **model** for a program $\mathcal{P}$ ($I \models \mathcal{P}$) **iff** $I\,\mathcal{P} = \top$
- This definition depends on the underlying logic!
  - We will indicate the underlying logic by adding a subscript to $\models$
  - Ł denotes Łukasiewicz logic
  - K denotes Kleene logic
  - F denotes Fitting logic
- Which of the following interpretations are models for

$$\mathcal{P} = \{a \leftarrow b\}$$

  - $\langle \emptyset, \emptyset \rangle \overset{?}{\models}_{Ł} \mathcal{P}$ $\quad \langle \{a, b\}, \emptyset \rangle \overset{?}{\models}_{Ł} \mathcal{P}$ $\quad \langle \emptyset, \{a, b\} \rangle \overset{?}{\models}_{Ł} \mathcal{P}$
  - $\langle \emptyset, \emptyset \rangle \overset{?}{\models}_{K} \mathcal{P}$ $\quad \langle \{a, b\}, \emptyset \rangle \overset{?}{\models}_{K} \mathcal{P}$ $\quad \langle \emptyset, \{a, b\} \rangle \overset{?}{\models}_{K} \mathcal{P}$
- In the sequel, we use Łukasiewicz logic if not stated otherwise

Model Intersection Property

- We would like to show that

$$\cap\{I \mid I \models \mathcal{P}\} \models \mathcal{P}$$

- It holds in classical two-valued logic for definite programs
- But it does not hold in classical two-valued logic for normal programs
- Under Łukasiewicz logic
  - The intersection of two models is not necessarily a model
  - Let $\mathcal{P}$ be the definite program

$$\begin{array}{lcl} p & \leftarrow & q_1 \wedge r_1 \\ p & \leftarrow & q_2 \wedge r_2 \end{array}$$

  - $\langle\emptyset, \{p, q_1, r_2\}\rangle \models \mathcal{P}$
  - $\langle\emptyset, \{p, q_2, r_1\}\rangle \models \mathcal{P}$
  - But $\langle\emptyset, \{p\}\rangle \not\models \mathcal{P}$

The Meaning of Programs

- **Proposition 10** If $I = \langle I^\top, I^\bot \rangle \models \mathcal{P}$ then $I' = \langle I^\top, \emptyset \rangle \models \mathcal{P}$
- **Proof** Suppose $I \models \mathcal{P}$,
  i.e., for all $A \leftarrow Body \in g\mathcal{P}$ we find $I \models A \leftarrow Body$
  - We consider the truth ordering $\bot <_t \mathsf{U} <_t \top$
  - We consider all cases for $I\,A$
  - We will show $I' \models A \leftarrow Body$ by $I'\,A \geq_t I'\,Body$
  - We distinguish three cases
    1. $I\,A = \top$ In this case $A \in I^\top$ and hence $I' \models A \leftarrow Body$
    2. $I\,A = \bot$
    3. $I\,A = \mathsf{U}$

Proof of Proposition 10 Case 2

2 $I\,A = \bot$ In this case $A \in I^{\bot}$ and $I'\,A = \mathrm{U}$

- ▷ Because $I \models A \leftarrow Body$ we conclude $I\,Body = \bot$
- ▷ Hence we find a literal $L \in Body$ such that $I\,L = \bot$
  - ▶▶ $L = B$ In this case $I\,B = \bot$ and hence $I'\,B = I'\,L = \mathrm{U}$
  - ▶▶ $L = \neg B$ In this case $I\,B = \top$ and hence $I'\,B = \top$ and $I'\,L = \bot$
- ▷ Consequently $I'\,Body \in \{\mathrm{U}, \bot\}$
- ▷ Because $I'\,A = \mathrm{U}$ we conclude $I' \models A \leftarrow Body$

Proof of Proposition 10 Case 3

3 $I\,A = \mathsf{U}$ In this case $I'\,A = \mathsf{U}$

- ▷ $I\,Body = \bot$ As in the previous case we find $I'\,Body \in \{\bot, \mathsf{U}\}$
  - ▶▶ Consequently $I' \models A \leftarrow Body$
- ▷ $I\,Body = \mathsf{U}$ In this case we find a literal $L \in Body$ with $I\,L = \mathsf{U}$
  - ▶▶ Then $I'\,L = \mathsf{U}$
  - ▶▶ Consequently $I'\,Body = \mathsf{U}$
  - ▶▶ Hence $I' \models A \leftarrow Body$ $\square$

Proposition 10 – Examples

- **Let** $\mathcal{P} = \{\ell \leftarrow e \wedge \neg ab_e,\ e \leftarrow \top,\ ab_e \leftarrow \bot\}$
  - $\langle\{e, \ell\}, \{ab_e\}\rangle \models \mathcal{P}$
  - $\langle\{e, \ell\}, \emptyset\rangle \models \mathcal{P}$
- **Let** $\mathcal{E} = \{a \approx b\}$ **and** $\mathcal{P} = \{q\,X \leftarrow \neg\,p\,X,\ p\,a \leftarrow \top\}$
  - $\langle\{[p\ a]\}, \{[q\ b]\}\rangle \models \mathcal{P}$
  - $\langle\{[p\ a]\}, \emptyset\rangle \models \mathcal{P}$
- **Does Proposition 10 hold under Kleene or Fitting logic?**

Intersection of Two Models with Empty $\bot$-Part

- **Proposition 11** Let $I_1 = \langle I_1^\top, \emptyset\rangle$ and $I_2 = \langle I_2^\top, \emptyset\rangle$ be two models of $\mathcal{P}$
  Then $I = \langle I_1^\top \cap I_2^\top, \emptyset\rangle$ is also a model of $\mathcal{P}$
- **Proof** Suppose $I \not\models \mathcal{P}$
  - Then we find $A \leftarrow Body \in g\mathcal{P}$ such that $I \not\models A \leftarrow Body$
  - We distinguish three cases
    1. $I\,A = \bot$ and $I\,Body = \top$ Impossible because $I^\bot = \emptyset$
    2. $I\,A = \bot$ and $I\,Body = \mathsf{U}$ Impossible because $I^\bot = \emptyset$
    3. $I\,A = \mathsf{U}$ and $I\,Body = \top$

       Because $I\,A = \mathsf{U}$ we find $j \in \{1, 2\}$ with $I_j\,A = \mathsf{U}$

       Because $I_j \models A \leftarrow Body$ we find $I_j\,Body \in \{\mathsf{U}, \bot\}$ $(*)$

       Because $I\,Body = \top$ and $I^\bot = \emptyset$ we find
       for all $L \in Body$ that L is an atom and $L \in I^\top$

       Hence for all $L \in Body$ we find $L \in I_j^\top$, $j \in \{1, 2\}$

       Consequently $I_j\,Body = \top$, $j \in \{1, 2\}$ contradicting $(*)$ $\square$

Model Intersection

- **Theorem 12** The model intersection property holds for $\mathcal{P}$
  i.e., $\cap\{I \mid I \models \mathcal{P}\} \models \mathcal{P}$
- **Proof** Follows immediately from Propositions 10 and 11 □
- **Example** Consider $\mathcal{P} = \{p \leftarrow q\}$
  - The least model of $\mathcal{P}$ under Łukasiewicz logic is $\langle \emptyset, \emptyset \rangle$
- Theorem 12 does not hold under Fitting logic ($\models_F$)
  - $\langle \{p, q\}, \emptyset \rangle \models_F p \leftarrow q$
  - $\langle \emptyset, \{p, q\} \rangle \models_F p \leftarrow q$
  - However $\langle \emptyset, \emptyset \rangle \not\models_F p \leftarrow q$
- Theorem 12 does not hold under Kleene logic ($\models_K$)
- What are the least models for the first three programs in the suppression task?

The Meaning of Weakly Completed Programs

- **Theorem 13** The model intersection property holds for $wc\,\mathcal{P}$ as well
- **Proof** later in the lecture
- $\mathcal{M}_{wc\mathcal{P}}$ denotes the least model of $wc\,\mathcal{P}$
- **Is $\mathcal{M}_{wc\mathcal{P}}$ the least model of $\mathcal{P}$?**
- **Corollary 14** If $I \models wc\,\mathcal{P}$ then $I \models \mathcal{P}$
- **Proof** $F \leftrightarrow G \equiv (F \rightarrow G) \wedge (G \rightarrow F)$ under Łukasiewicz logic $\square$
- Proposition 14 does not hold under Fitting logic
  - $\langle \emptyset, \emptyset \rangle \models_F wc\{p \leftarrow q\} = \{p \leftrightarrow q\}$
  - However $\langle \emptyset, \emptyset \rangle \not\models_F \{p \leftarrow q\}$

The Suppression Task – Experiments 1-3

Ex.	$\mathcal{P}$	$wc\,\mathcal{P}$	$\mathcal{M}_{wc\mathcal{P}}$
1	$e \leftarrow \top$ $\ell \leftarrow e \wedge \neg ab_e$ $ab_e \leftarrow \bot$	$e \leftrightarrow \top$ $\ell \leftrightarrow e \wedge \neg ab_e$ $ab_e \leftrightarrow \bot$	$\langle \{e, \ell\}, \{ab_e\} \rangle$
2	$e \leftarrow \top$ $\ell \leftarrow e \wedge \neg ab_e$ $\ell \leftarrow t \wedge \neg ab_t$ $ab_e \leftarrow \bot$ $ab_t \leftarrow \bot$	$e \leftrightarrow \top$ $\ell \leftrightarrow (e \wedge \neg ab_e) \vee (t \wedge \neg ab_t)$ $ab_e \leftrightarrow \bot$ $ab_t \leftrightarrow \bot$	$\langle \{e, \ell\}, \{ab_e, ab_t\} \rangle$
3	$e \leftarrow \top$ $\ell \leftarrow e \wedge \neg ab_e$ $\ell \leftarrow o \wedge \neg ab_o$ $ab_e \leftarrow \bot$ $ab_o \leftarrow \bot$ $ab_e \leftarrow \neg o$ $ab_o \leftarrow \neg e$	$e \leftrightarrow \top$ $\ell \leftrightarrow (e \wedge \neg ab_e) \vee (o \wedge \neg ab_o)$ $ab_e \leftrightarrow \bot \vee \neg o$ $ab_o \leftrightarrow \bot \vee \neg e$	$\langle \{e\}, \{ab_o\} \rangle$

The Suppression Task – Experiments 4-6

Ex.	$\mathcal{P}$	$wc\,\mathcal{P}$	$\mathcal{M}_{wc\mathcal{P}}$
4	$e \leftarrow \bot$ $\ell \leftarrow e \wedge \neg ab_e$ $ab_e \leftarrow \bot$	$e \leftrightarrow \bot$ $\ell \leftrightarrow e \wedge \neg ab_e$ $ab_e \leftrightarrow \bot$	$\langle \emptyset, \{e, \ell, ab_e\} \rangle$
5	$e \leftarrow \bot$ $\ell \leftarrow e \wedge \neg ab_e$ $\ell \leftarrow t \wedge \neg ab_t$ $ab_e \leftarrow \bot$ $ab_t \leftarrow \bot$	$e \leftrightarrow \bot$ $\ell \leftrightarrow (e \wedge \neg ab_e) \vee (t \wedge \neg ab_t)$ $ab_e \leftrightarrow \bot$ $ab_t \leftrightarrow \bot$	$\langle \emptyset, \{e, ab_e, ab_t\} \rangle$
6	$e \leftarrow \bot$ $\ell \leftarrow e \wedge \neg ab_e$ $\ell \leftarrow o \wedge \neg ab_o$ $ab_e \leftarrow \bot$ $ab_o \leftarrow \bot$ $ab_e \leftarrow \neg o$ $ab_o \leftarrow \neg e$	$e \leftrightarrow \bot$ $\ell \leftrightarrow (e \wedge \neg ab_e) \vee (o \wedge \neg ab_o)$ $ab_e \leftrightarrow \bot \vee \neg o$ $ab_o \leftrightarrow \bot \vee \neg e$	$\langle \{ab_o\}, \{e, \ell\} \rangle$

Monotonicity

- Let $\mathcal{P}$ and $\mathcal{P}'$ be sets of formulas and G a formula
  A logic is monotonic if the following holds:
  If $\mathcal{P} \models G$ then $\mathcal{P} \cup \mathcal{P}' \models G$
- Classical logic is monotonic
- A logic based on the weak completion semantics is non-monotonic
  - Consider

$$\begin{array}{lcl} \mathcal{P} & = & \{c \leftarrow \bot\} \\ \mathcal{P}' & = & \{c \leftarrow \top\} \end{array}$$

  - Then

$$\begin{array}{lclcl} wc\,\mathcal{P} & = & \{c \leftrightarrow \bot\} & \models & \neg c \\ wc\,(\mathcal{P} \cup \mathcal{P}') & = & \{c \leftrightarrow \bot \vee \top\} & \models & c \end{array}$$

Computing Least Models

- **How can we compute the least models of weakly completed programs?**
- **In classical two-valued logic we obtain**

$$T_{\mathcal{P}}\, I = \{A \mid \text{there exists } A \leftarrow Body \in g\,\mathcal{P} \text{ with } I\, Body = \top\}$$

where $\mathcal{P}$ is a definite logic program and I an interpretation

- **In three-valued logic programming we obtain $\Psi_{\mathcal{P}}\, I = \langle J^\top, J^\bot \rangle$ where**

$$\begin{array}{lll} J^\top & = & \{A \mid \text{there exists } A \leftarrow Body \in g\,\mathcal{P} \text{ with } I\, Body = \top\} \\ J^\bot & = & \{A \mid \text{for all } A \leftarrow Body \in g\,\mathcal{P} \text{ we find } I\, Body = \bot\} \end{array}$$

  - **$\Psi_{\mathcal{P}}$ is monotone on $(\mathcal{I}, \subseteq)$**
  - **The least model of $c\,\mathcal{P}$ under Fitting logic is the least fixed point of $\Psi_{\mathcal{P}}$**
  - **Inadequate for human reasoning $\rightsquigarrow$ Why?**

The Semantic Operator for Weakly Completed Programs

- Consider the following immediate consequence operator
  $\Phi'_{\mathcal{P}}\, I = \langle J^\top, J^\bot \rangle$ where

$$\begin{array}{lcl} J^\top & = & \{A \mid \text{there exists } A \leftarrow Body \in g\,\mathcal{P} \text{ with } I\,Body = \top\} \\ J^\bot & = & \{A \mid \text{there exists } A \leftarrow Body \in g\,\mathcal{P} \text{ and} \\ & & \quad \text{for all } A \leftarrow Body \in g\,\mathcal{P} \text{ we find } I\,Body = \bot\} \end{array}$$

- $\Phi'_{\mathcal{P}}$ “without the red condition” is $\Psi_{\mathcal{P}}$

The Semantic Operator for Weakly Completed Programs with Equality

- Let $\mathcal{P}$ be a program, $\mathcal{E}$ an equational theory, and I an interpretation
- Consider the following immediate consequence operator
  $\Phi_{\mathcal{P}}\, I = \langle J^{\top}, J^{\perp} \rangle$ where

$$
\begin{array}{lcl}
J^{\top} & = & \{[A] \mid \text{there exists } A \leftarrow Body \in g\,\mathcal{P} \text{ with } I\, Body = \top\} \\
J^{\perp} & = & \{[A] \mid \text{there exists } A \leftarrow Body \in g\,\mathcal{P} \text{ and} \\
 & & \quad \text{for all } A' \leftarrow Body \in g\,\mathcal{P} \text{ with } [A] = [A'] \text{ we find } I\, Body = \bot\}
\end{array}
$$

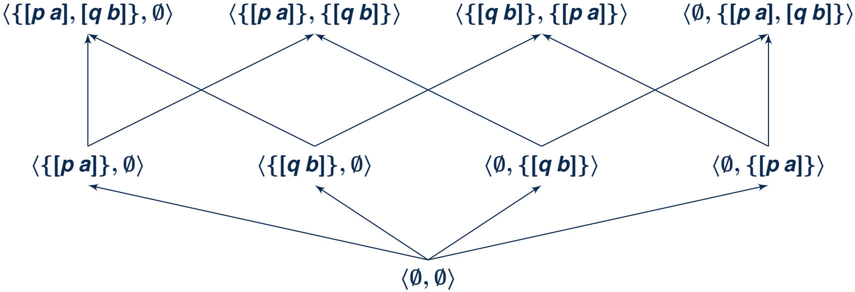
  and $[A]$ denotes the finest congruence class defined by $\mathcal{E}$ and containing A

Semantic Operator – Examples

- **Iteratively apply $\Phi_{\mathcal{P}}$ to the following programs starting with $\langle \emptyset, \emptyset \rangle$**
  - $\mathcal{P} = \{e \leftarrow \top,\ \ell \leftarrow e \wedge \neg ab_e,\ ab_e \leftarrow \bot\}$ **and** $\mathcal{E} = \emptyset$
  - $\mathcal{P} = \{q\,X \leftarrow \neg p\,X,\ p\,a \leftarrow \top\}$ **and** $\mathcal{E} = \{a \approx b\}$
- **Do least fixed points of $\Phi_{\mathcal{P}}$ always exist?**
- **How long does it take to compute least fixed points of $\Phi_{\mathcal{P}}$?**
  - **Recall fixed point theory**

The Complete Partial Order of Interpretations – Example

- Let $\mathcal{P} = \{p\,X \leftarrow q\,X\}$ and $\mathcal{E} = \{a \approx b\}$
- Let $\mathcal{I}$ denote the set of all three-valued interpretations
- $I = \langle I^\top, I^\bot \rangle \subseteq \langle J^\top, J^\bot \rangle = J$ iff $I^\top \subseteq J^\top$ and $I^\bot \subseteq J^\bot$
- $(\mathcal{I}, \subseteq)$ is a complete partial order



The Complete Partial Order of Interpretations 1

- Let $\mathcal{P}$ be a program and $\mathcal{E}$ an equational theory
- Let $\mathcal{J}$ be a set of interpretations
  - $\mathcal{J}^\top = \{I^\top \mid \langle I^\top, I^\bot\rangle \in \mathcal{J}\}$
  - $\mathcal{J}^\bot = \{I^\bot \mid \langle I^\top, I^\bot\rangle \in \mathcal{J}\}$
- **Proposition 15** Let $\mathcal{J}$ be a directed set of interpretations
  Then the interpretation $I = \langle\bigcup \mathcal{J}^\top, \bigcup \mathcal{J}^\bot\rangle$ is the least upper bound of $\mathcal{J}$
- **Proof** Given $\mathcal{J}$ we have to show that

  (i) I is an interpretation

  (ii) I is an upper bound of $\mathcal{J}$ $\rightsquigarrow$ Exercise

  (iii) I is the least upper bound of $\mathcal{J}$ $\rightsquigarrow$ Exercise

Proof of Proposition 15 (i)

- **To show** $I = \langle \bigcup \mathcal{J}^\top, \bigcup \mathcal{J}^\bot \rangle$ is an interpretation
  - By definition $\bigcup \mathcal{J}^\top$ and $\bigcup \mathcal{J}^\bot$ are unions of congruence classes
  - It remains to show $\bigcup \mathcal{J}^\top \cap \bigcup \mathcal{J}^\bot = \emptyset$
  - Suppose we find $[A] \in \bigcup \mathcal{J}^\top \cap \bigcup \mathcal{J}^\bot$
  - Then we find $I_1, I_2 \in \mathcal{J}$ with $[A] \in I_1^\top$ and $[A] \in I_2^\bot$
  - Because $\mathcal{J}$ is directed, it contains a common upper bound K of I_1 and I_2
  - We find $[A] \in K^\top$ and $[A] \in K^\bot$
  - Hence, K cannot be an interpretation $\rightsquigarrow$ contradiction $\square$

The Complete Partial Order of Interpretations 2

- **Corollary 16**
  The set of all interpretations $\mathcal{I}$ is a complete partial order with respect to $\subseteq$
- **Proof**
  - Reflexivity, antisymmetry, and transitivity holds for $\subseteq$
  - By Proposition 15 every directed subset of $\mathcal{I}$ has a least upper bound in $\mathcal{I}$

Monotonocity of the Semantic Operator

- **Proposition 17**
  For each program $\mathcal{P}$ and equational theory $\mathcal{E}$ the mapping $\Phi_{\mathcal{P}}$ is monotonic
- **Proof** Let $I = \langle I^\top, I^\bot \rangle \subseteq \langle J^\top, J^\bot \rangle = J$
  - **To show** $\Phi_{\mathcal{P}}\, I = I' = \langle I'^\top, I'^\bot \rangle \subseteq \langle J'^\top, J'^\bot \rangle = J' = \Phi_{\mathcal{P}}\, J$
  - $I'^\top \subseteq J'^\top$
    - $[A] \in I'^\top$ iff we find $A \leftarrow Body \in g\,\mathcal{P}$ such that $I\ Body = \top$
    - Because $I \subseteq J$ we claim $J\ Body = \top$ prove it!
    - Hence, $[A] \in J'^\top$
  - $I'^\bot \subseteq J'^\bot$ $\rightsquigarrow$ Exercise □

Non-Continuity of the Semantic Operator 1

- Let $\mathcal{E} = \emptyset$ and $\mathcal{P}$ be

$$\begin{array}{rcl} q\,a & \leftarrow & \top \\ q\,s\,X & \leftarrow & q\,X \\ p & \leftarrow & \neg\, q\,X \end{array}$$

- The least fixed point of $\Phi_{\mathcal{P}}$ is

$$\langle \{[q\,s^k\,a] \mid k \in \mathbb{N}\}, \{[p]\} \rangle$$

- It is reached after $\omega + 1$ iterations
- By the Kleene Fixed Point Theorem 4 $\Phi_{\mathcal{P}}$ is not continuous
- The Herbrand base contains infinitely many equivalence classes

$$[p],\ [q\,a],\ [q\,s\,a],\ \ldots$$

where each equivalence class has one member

Non-Continuity of the Semantic Operator 2

- Let $\mathcal{P}$ be

$$\begin{array}{rcl} q\ 1 & \leftarrow & \top \\ q\ (X \circ a) & \leftarrow & q\ X \\ p & \leftarrow & \neg\, q\ X \end{array}$$

and $\mathcal{E}$ be

$$\begin{array}{rcl} X \circ (Y \circ Z) & \approx & (X \circ Y) \circ Z \\ X \circ Y & \approx & Y \circ X \\ X \circ 1 & \approx & X \end{array}$$

- The least fixed point of $\Phi_{\mathcal{P}}$ is

$$\langle \{ [q(1 \circ \overbrace{a \circ \ldots \circ a}^{k})] \mid k \in \mathbb{N} \}, \{[p]\} \rangle$$

- It is reached after $\omega + 1$ iterations
- By Kleene Fixed Point Theorem 4 $\Phi_{\mathcal{P}}$ is not continuous
- The Herbrand base contains infinitely many equivalence classes

$$[p],\ [q\ 1],\ [q\ a],\ [q(a \circ a)],\ \ldots$$

where with the exception of $[p]$ each of these equivalence classes is infinite

Finite Propositional and Finite Ground Programs

- **Proposition 18**
  For each finite propositional program $\mathcal{P}$ the mapping $\Phi_{\mathcal{P}}$ is continuous
- **Proof**
  - Because $\mathcal{P}$ is finite, the set $\mathcal{I}$ of interpretations is finite
  - By Corollary 16 $(\mathcal{I}, \subseteq)$ is a complete partial order
  - By Proposition 17 $\Phi_{\mathcal{P}}$ is monotonic on $\mathcal{I}$
  - By Proposition 7 the mapping $\Phi_{\mathcal{P}}$ is continuous □
- **Proposition 19**
  If the Herbrand base for a program $\mathcal{P}$ and a set of equations $\mathcal{E}$ is finite then the mapping $\Phi_{\mathcal{P}}$ is continuous
- **Proof**
  - Define a bijection between the set of ground atoms occurring in $\mathcal{P}$ and an equally large set of propositional atoms
  - Replace each ground atom by a propositional atom
  - Apply Proposition 18 □

Least Fixed Points and Models

- **Lemma 20** Let J be the least fixed point of $\Phi_{\mathcal{P}}$ and I a model of $wc\mathcal{P}$
  - Then for every ground atom A we find
    - If $J\,A = \top$ then $I\,A = \top$
    - If $J\,A = \bot$ then $I\,A = \bot$
- **Proof** Let J be the least fixed point of $\Phi_{\mathcal{P}}$ and I a model of $wc\mathcal{P}$
  - We start iterating $\Phi_{\mathcal{P}}$ on $\langle\emptyset, \emptyset\rangle$
  - **Claim** For every ordinal α and every ground atom A we find
    - If $\Phi_{\mathcal{P}} \uparrow \alpha\; A = \top$ then $I\,A = \top$
    - If $\Phi_{\mathcal{P}} \uparrow \alpha\; A = \bot$ then $I\,A = \bot$
  - **Proof of the Claim** by transfinite induction $\rightsquigarrow$ **Exercise**
  - The lemma follows from Propositions 3 and 17 $\square$

Lemma 20 – Example

- Let $\mathcal{P} = \{q\,a \leftarrow \top,\ q\,s\,X \leftarrow q\,X,\ p \leftarrow \neg\,q\,X,\ r\,a \leftarrow \top\}$
- $I = \langle \{q\,s^k\,a \mid k \in \mathbb{N}\} \cup \{r\,a,\ r\,s^2\,a\}, \{p,\ r\,s\,a\}\rangle$ is a model of $wc\,\mathcal{P}$

$$
\begin{array}{rl}
\Phi_{\mathcal{P}} \uparrow 0 & \langle \emptyset, \emptyset \rangle \\
\Phi_{\mathcal{P}} \uparrow 1 & \langle \{q\,a,\ r\,a\}, \emptyset \rangle \\
\Phi_{\mathcal{P}} \uparrow 2 & \langle \{q\,a,\ q\,s\,a,\ r\,a\}, \emptyset \rangle \\
\vdots & \vdots \\
\Phi_{\mathcal{P}} \uparrow \omega & \langle \{q\,s^k\,a \mid k \in \mathbb{N}\} \cup \{r\,a\}, \emptyset \rangle \\
\Phi_{\mathcal{P}} \uparrow (\omega + 1) & \langle \{q\,s^k\,a \mid k \in \mathbb{N}\} \cup \{r\,a\}, \{p\} \rangle
\end{array}
$$

Fixed Points are Models

- **Lemma 21**
  If I is a fixed point of $\Phi_{\mathcal{P}}$ then I is a model of $wc\,\mathcal{P}$
- **Proof** to show $I(A \leftrightarrow F) = \top$ for all $A \leftrightarrow F \in wc\,\mathcal{P}$
  - $[A] \in I^\top$ We find $A \leftarrow Body \in \mathcal{P}$ with $I\,Body = \top$
    - Then, $F = Body \vee F'$ and $I\,F = \top$
    - Hence, $I\,A = I\,F$
  - $[A] \in I^\bot$ $\rightsquigarrow$ Exercise
  - $[A] \not\in I^\top \cup I^\bot$ $\rightsquigarrow$ Exercise $\square$

Least Fixed Points are Minimal Models

- **Proposition 22**
  If J is the least fixed point of $\Phi_{\mathcal{P}}$ then J is a minimal model of $wc\,\mathcal{P}$
- **Proof** Let J be the least fixed point of $\Phi_{\mathcal{P}}$
  - By Lemma 21 J is a model of $wc\,\mathcal{P}$
  - By Proposition 20 for every model I of $wc\mathcal{P}$ we find $J^\top \subseteq I^\top$ and $J^\bot \subseteq I^\bot$, i.e., $J \subseteq I$
  - Hence, J is a minimal model of $wc\,\mathcal{P}$ $\square$

Least Fixed Points and Least Models

- **Proposition 23**
  If I is a minimal model of $wc\,\mathcal{P}$ then I is the least fixed point of $\Phi_{\mathcal{P}}$
- **Proof** Let I be a minimal model of $wc\mathcal{P}$ and J be the least fixed point of $\Phi_{\mathcal{P}}$
  - From Lemma 20 we learn that $J^\top \subseteq I^\top$ and $J^\perp \subseteq I^\perp$
  - But then $I = J$ as otherwise we have a conflict with the minimality of I □
- **Theorem 13** $wc\,\mathcal{P}$ has a least model
- **Proof** Follows from Propositions 22 and 23 and the fact that the least fixed point of $\Phi_{\mathcal{P}}$ is unique □
- **Theorem 24** I is the least fixed point of $\Phi_{\mathcal{P}}$ iff I is the least model of $wc\,\mathcal{P}$
- **Proof** Follows from Theorem 13 and Propositions 22 and 23 □

Entailment under the Weak Completion Semantics

- Let $\mathcal{M}_{wc\mathcal{P}}$ denote the least fixed point of $\Phi_{\mathcal{P}}$
  - which is equal to the least model of $wc\,\mathcal{P}$
- $\mathcal{P}$ entails F under the weak completion semantics

$$\mathcal{P} \models_{wcs} F \quad \text{iff} \quad \mathcal{M}_{wc\mathcal{P}}\, F = \top$$

Two Examples

- Consider the program $\mathcal{P} = \{p \leftarrow q,\ q \leftarrow p\}$
  - It has a least model $\langle \emptyset, \emptyset \rangle$
  - It can be computed iterating $\Phi_{\mathcal{P}}$ starting with $\langle \emptyset, \emptyset \rangle$
  - But if the iteration would start with $\langle \{p\}, \emptyset \rangle$ then it will run forever
  - Do humans always start with the empty interpretation?
- Consider the program $\mathcal{P} = \{even\ 0 \leftarrow \top,\ even\ s\,X \leftarrow \neg\, even\ X\}$
  - It has a least model $\langle \{even\ s^k\ 0 \mid k \text{ is even}\}, \{even\ s^k\ 0 \mid k \text{ is odd}\} \rangle$
  - It can be computed iterating $\Phi_{\mathcal{P}}$ starting with $\langle \emptyset, \emptyset \rangle$
  - How many steps do we need?
- We will address both questions using metric methods

Semantic Operators as Contraction Mappings

- A level mapping for $\mathcal{P}$ is a mapping *level* from the set of ground atoms to $\mathbb{N}$ such that $level\ A = level\ B$ iff $[A] = [B]$
  - It is extended to a mapping from ground literals to $\mathbb{N}$ by $level\ \neg A = level\ A$
- Let *level* be a level mapping for $\mathcal{P}$
  - $\mathcal{P}$ is acyclic with respect to *level* iff
    for every rule $A \leftarrow L_1 \wedge \ldots \wedge L_n \in g\,\mathcal{P}$
    we find $level\ A > level\ L_i$ for all $1 \leq i \leq n$
  - $\mathcal{P}$ is acyclic iff it is acyclic with respect to some level mapping
  - The problem to determine whether $\mathcal{P}$ is acyclic is undecidable

Acyclic Programs – Examples 1

- Consider the program $\mathcal{P}$

$$\begin{array}{lcl} p & \leftarrow & r \wedge q \\ q & \leftarrow & r \wedge p \end{array}$$

  - Is $\mathcal{P}$ acyclic?
  - How many fixed points has $\Phi_{\mathcal{P}}$?
  - Is $\Phi_{\mathcal{P}}$ a contraction on a complete metric space?

- Are the followig programs acyclic?
  - $\{q\, a \leftarrow \top,\; q\, s\, X \leftarrow q\, X,\; p \leftarrow \neg\, q\, X\}$
  - $\{even\ 0 \leftarrow \top,\; even\ s\, X \leftarrow \neg\, even\ X\}$

Acyclic Programs – Examples 2

- Consider the program $\mathcal{P}$

$$\begin{array}{lcl} p & \leftarrow & q \wedge r \\ q & \leftarrow & \neg r \\ r & \leftarrow & \top \end{array}$$

- Let *level* $r = 0$, *level* $q = 1$, *level* $p = 2$
  - $\mathcal{P}$ is acyclic with respect to *level*
  - We find

$$\begin{array}{lcl} \Phi_{\mathcal{P}}(\langle\{q,r\},\{p\}\rangle) & = & \langle\{p,r\},\{q\}\rangle \\ \Phi_{\mathcal{P}}(\langle\{p,r\},\{q\}\rangle) & = & \langle\{r\},\{p,q\}\rangle \end{array}$$

$$\begin{array}{lcl} \Phi_{\mathcal{P}}(\langle\{p\},\emptyset\rangle) & = & \langle\{r\},\emptyset\rangle \\ \Phi_{\mathcal{P}}(\langle\{r\},\emptyset\rangle) & = & \langle\{r\},\{q\}\rangle \\ \Phi_{\mathcal{P}}(\langle\{r\},\{q\}\rangle) & = & \langle\{r\},\{p,q\}\rangle \end{array}$$

  - $\langle\{r\},\{p,q\}\rangle$ is the unique fixed point of $\Phi_{\mathcal{P}}$
  - Is $\Phi_{\mathcal{P}}$ a contraction? If so, on what metric space?

Programs and Metric Spaces

- **Proposition 25** Let $\mathcal{P}$ be a program, $\mathcal{E}$ an equational theory, *level* a level mapping for $\mathcal{P}$, $\mathcal{I}$ the set of interpretations for $\mathcal{P}$, and $I, J \in \mathcal{I}$
  - The function $d_{level} : \mathcal{I} \times \mathcal{I} \to \mathbb{R}$ defined as

$$d_{level}(I, J) = \begin{cases} \frac{1}{2^n} & I \neq J \text{ and} \\ & I\ A = J\ A \neq \mathsf{U} \text{ for all } A \text{ with } level\ A < n \text{ and} \\ & I\ A \neq J\ A \text{ or } I\ A = J\ A = \mathsf{U} \text{ for some } A \text{ with } level\ A = n \\ 0 & \text{otherwise} \end{cases}$$

  is a metric

- **Proof** $\rightsquigarrow$ **Exercise**

Programs and Metric Spaces – Example 1

- **Consider the program $\mathcal{P}$**

$$\begin{array}{rcl} \textit{even } 0 & \leftarrow & \top \\ \textit{even } s\, X & \leftarrow & \neg\, \textit{even } X \end{array}$$

- **Let**

$$\begin{array}{rcl} I & = & \langle\{\textit{even } s^k\, 0 \mid k \in \{0, 2, \ldots\}\}, \{\textit{even } s^k\, 0 \mid k \in \{1, 3, \ldots\}\}\rangle \\ J & = & \langle\{\textit{even } s^k\, 0 \mid k \in \{0, 2, \ldots\}\}, \emptyset\rangle \end{array}$$

and

$$\textit{level even } s^k\, 0 = k$$

- **Then**

$$d_{\textit{level}}(I, J) = \frac{1}{2}$$

- **Note** $g\,\mathcal{P}$ **is infinite and** $\mathcal{P}$ **is acyclic**

Programs and Metric Spaces – Example 2

- Consider again the program $\mathcal{P}$

$$\begin{array}{rcl} \textit{even } 0 & \leftarrow & \top \\ \textit{even } s\,X & \leftarrow & \neg\, \textit{even } X \end{array}$$

- Let again $\textit{level even } s^k\ 0 = k$
- For all $n \in \mathbb{N}$ let

$$I_n = \langle \{\textit{even } s^k\ 0 \mid k \leq n \text{ and } k \text{ even}\}, \{\textit{even } s^k\ 0 \mid k \leq n \text{ and } k \text{ odd}\} \rangle$$

- What is the distance between I_n and I_m?
- Is the sequence $(I_n \mid n \geq 0)$ a Cauchy sequence?
- Does the sequence $(I_n \mid n \geq 0)$ converge?

Programs and Complete Metric Spaces

- Let *level* be a level mapping for $\mathcal{P}$, $\mathcal{E}$ an equational theory and $\mathcal{I}$ the set of interpretations for $\mathcal{P}$
- **Proposition 26** $(\mathcal{I}, d_{level})$ is a complete metric space
- **Proof** **To show** Every Cauchy sequence of interpretations converges
  - Let $(I_k \mid k \geq 1)$ be a Cauchy sequence of interpretations
  - I.e., for all $\varepsilon > 0$ there is $K \in \mathbb{N}$: for all $k_1, k_2 \geq K$ we find $d_{level}(I_{k_1}, I_{k_2}) \leq \varepsilon$
  - In particular, for all $n \in \mathbb{N}$, there is $K \in \mathbb{N}$: for all $k_1, k_2 \geq K$ we find

$$d_{level}(I_{k_1}, I_{k_2}) \leq \frac{1}{2^{n+1}}$$

  - For all $n \in \mathbb{N}$ let K_n be the least such K
  - Hence, if $n_1 \leq n_2$ then $\frac{1}{2^{n_1+1}} \geq \frac{1}{2^{n_2+1}}$ and $K_{n_1} \leq K_{n_2}$
  - **To show** $(I_k \mid k \geq 1)$ converges
  - i.e., there is I: for every $\varepsilon > 0$, there is a K: for all $k \geq K$ we find $d(I, I_k) \leq \varepsilon$

Proof of Proposition 26 – Continued

- Let I be such that for each ground atom A we have $I\,A = I_{K_\ell}\,A$ where $\ell = level\,A$
- We choose $\varepsilon > 0$ and let $n \in \mathbb{N}$ be such that $\frac{1}{2^{n+1}} \leq \varepsilon$
- **Claim** $d_{level}(I, I_k) \leq \frac{1}{2^{n+1}} \leq \varepsilon$ for any $k \geq K_n$
- **Proof of the Claim** $\rightsquigarrow$ **Exercise** $\square$

Programs and Contractions

- **Let *level* be a level mapping for $\mathcal{P}$, $\mathcal{E}$ an equational theory and $\mathcal{I}$ the set of interpretations for $\mathcal{P}$**
- **Theorem 27**
  If $\mathcal{P}$ is acyclic with respect to *level* then $\Phi_{\mathcal{P}}$ is a contraction on $(\mathcal{I}, d_{level})$
- **Proof** **we will prove a more general result later in the lecture**
- **Corollary 28** **If $\mathcal{P}$ is acyclic then $\Phi_{\mathcal{P}}$ has a unique fixed point which can be reached by iterating $\Phi_{\mathcal{P}}$ up to ω times starting with any interpretation**
- **Proof** **Follows from Theorems 27 and 9** □

Reconsidering Two Examples

- Reconsider the program $\mathcal{P} = \{p \leftarrow q,\ q \leftarrow p\}$
  - It is not acyclic
  - Model construction must start with the empty interpretation
- Reconsider the program $\mathcal{P} = \{even\ 0 \leftarrow \top,\ even\ s\ X \leftarrow \neg\ even\ X\}$
  - It is acyclic
  - Model construction can start with any interpretation

$\Phi_{\mathcal{P}}$	$I^{\top}$	$I^{\bot}$
$\uparrow 0$		*even 0*
$\uparrow 1$	*even 0* *even s 0*	
$\uparrow 2$	*even 0*	*even s 0* *even s s 0*
$\vdots$	$\vdots$	$\vdots$

  - The least fixed point will be computed in ω steps

Abduction – Overview

- **Integrity constraints**
- **Abducibles**
- **Abductive Frameworks**
- **Observations**
- **Credulous versus skeptical reasoning**
- **Examples**

Abduction

- Charles Sanders Peirce 1932
  - given a program and an observation (which is not entailed by the program)
  - a consistent set of facts (and assumptions) is infered or **abduced**
  - such that the program and the facts entail the observation
- The set of facts is called **explanation** for the observation
- **Applications**
  - fault diagnosis
  - high level vision
  - natural language processing
  - planning
  - knowledge assimilation
  - . . .

Integrity Constraints

- **Integrity constraints are formulas of the form**

$$\mathsf{U} \leftarrow \textit{Body} \text{ (weak IC)} \quad \text{or} \quad \bot \leftarrow \textit{Body} \text{ (strong IC)}$$

where *Body* is a conjunction of literals

- $\mathcal{IC}$ **denotes a finite set of integrity constraints**
- **Interpretation I satisfies $\mathcal{IC}$ iff I satisfies each constraint occurring in $\mathcal{IC}$**
- **Integrity constraints eliminate models**
- **Examples**

a	$\mathsf{U} \leftarrow a$	$\bot \leftarrow a$	$\mathsf{U} \leftarrow \neg a$	$\bot \leftarrow \neg a$
$\top$	U	$\bot$	$\top$	$\top$
U	$\top$	U	$\top$	U
$\bot$	$\top$	$\top$	U	$\bot$

- **What is the difference between $\bot \leftarrow a$ and $a \leftarrow \bot$?**

Integrity Constraints – Preferences

- **Michael believes that offering Kim a homemade cake or homemade cookies will make her happy. But he also knows that she does not want both.**

$$
\begin{array}{rcl}
happy & \leftarrow & cake \wedge \neg ab_{cake} \\
happy & \leftarrow & cookies \wedge \neg ab_{cookies} \\
ab_{cake} & \leftarrow & \bot \\
ab_{cookies} & \leftarrow & \bot
\end{array}
$$

cake	*cookies*	$U \leftarrow cake \wedge cookies$	$\bot \leftarrow cake \wedge cookies$
$\top$	$\top$	U	$\bot$
$\top$	U	$\top$	U
$\top$	$\bot$	$\top$	$\top$
U	$\top$	$\top$	U
U	U	$\top$	U
U	$\bot$	$\top$	$\top$
$\bot$	$\top$	$\top$	$\top$
$\bot$	U	$\top$	$\top$
$\bot$	$\bot$	$\top$	$\top$

Integrity Constraints and Models

- **Suppose $\mathcal{IC} \neq \emptyset$**
- **Then $\mathcal{P}$ as well as *wc* $\mathcal{P}$ may not have models satisfying $\mathcal{IC}$**
- **Can you specify an example?**

Abducibles

- Let $\mathcal{P}$ be a ground program
- The **set of abducibles** is

$$\mathcal{A}_{\mathcal{P}} = \{A \leftarrow \top \mid A \text{ is undefined in } \mathcal{P}\} \cup \{A \leftarrow \bot \mid A \text{ is undefined in } \mathcal{P}\}$$

- **Should defeaters of negative assumptions be added to this set?**

Abductive Frameworks

- Let $\mathcal{P}$ be a ground program
- An abductive framework $\langle \mathcal{P}, \mathcal{A}_\mathcal{P}, \mathcal{IC}, \models_{wcs} \rangle$ consists of
  - a program $\mathcal{P}$
  - a set of abducibles $\mathcal{A}_\mathcal{P}$
  - a set $\mathcal{IC}$ of integrity constraints
  - the entailment relation $\models_{wcs}$
- In the sequel, we sometimes consider datalog programs
  - In this case, the set of abducibles as well as abductive frameworks are defined with respect to the ground instances of the program

The Suppression Task – Abducibles

$\mathcal{P}$			$\mathcal{A}_{\mathcal{P}}$		
ℓ	$\leftarrow$	$e \wedge \neg ab_e$	e	$\leftarrow$	$\top$
ab_e	$\leftarrow$	$\bot$	e	$\leftarrow$	$\bot$
ℓ	$\leftarrow$	$e \wedge \neg ab_e$	e	$\leftarrow$	$\top$
ℓ	$\leftarrow$	$t \wedge \neg ab_t$	e	$\leftarrow$	$\bot$
ab_e	$\leftarrow$	$\bot$	t	$\leftarrow$	$\top$
ab_t	$\leftarrow$	$\bot$	t	$\leftarrow$	$\bot$
ℓ	$\leftarrow$	$e \wedge \neg ab_e$	e	$\leftarrow$	$\top$
ℓ	$\leftarrow$	$o \wedge \neg ab_o$	e	$\leftarrow$	$\bot$
ab_e	$\leftarrow$	$\bot$	o	$\leftarrow$	$\top$
ab_o	$\leftarrow$	$\bot$	o	$\leftarrow$	$\bot$
ab_e	$\leftarrow$	$\neg o$			
ab_o	$\leftarrow$	$\neg e$			

Observations and Explanations

- **An observation $\mathcal{O}$ is a set of ground literals**
- **$\mathcal{O}$ is explainable in the abductive framework $\langle \mathcal{P}, \mathcal{A_P}, \mathcal{IC}, \models_{wcs} \rangle$**
  iff there exists a non-empty $\mathcal{X} \subseteq \mathcal{A_P}$ called explanation such that
  - $\mathcal{M}_{wc(\mathcal{P} \cup \mathcal{X})} \models_{wcs} L$ **for all** $L \in \mathcal{O}$
  - $\mathcal{M}_{wc(\mathcal{P} \cup \mathcal{X})}$ **satisfies** $\mathcal{IC}$
- **Sometimes explanations are required to be minimal**
  - **where $\mathcal{X}$ is minimal if there does not exist an explanation $\mathcal{X}'$ with $\mathcal{X}' \subseteq \mathcal{X}$**
- **Is $\mathcal{P} \cup \mathcal{X}$ satisfiable?**
- **Is the empty observation explainable?**

Observations and Explanations – Example

- Let $\mathcal{P}$ consist of

$$\begin{array}{lcl} happy & \leftarrow & cake \wedge \neg ab_{cake} \\ happy & \leftarrow & cookies \wedge \neg ab_{cookies} \\ ab_{cake} & \leftarrow & \bot \\ ab_{cookies} & \leftarrow & \bot \end{array}$$

- Then $\mathcal{A}_\mathcal{P}$ consists of

$$\begin{array}{lclclcl} cake & \leftarrow & \top & \quad & cookies & \leftarrow & \top \\ cake & \leftarrow & \bot & \quad & cookies & \leftarrow & \bot \end{array}$$

- Let $\mathcal{IC} = \{\mathsf{U} \leftarrow cake \wedge cookies\}$
- Let $\mathcal{O} = \{happy\}$
- $\{cake \leftarrow \top\}$ and $\{cookies \leftarrow \top\}$ are explanations
- $\{cake \leftarrow \top,\ cookies \leftarrow \top\}$ is not an explanation

The Suppression Task – Experiments 7-9

Ex.	$\mathcal{P}$	$\mathcal{A}_\mathcal{P}$	$\mathcal{O}$	$\mathcal{X}$		e
7	$\ell \leftarrow e \wedge \neg ab_e$ $ab_e \leftarrow \bot$	$e \leftarrow \top$ $e \leftarrow \bot$	ℓ	$e \leftarrow \top$		0.71
8	$\ell \leftarrow e \wedge \neg ab_e$ $\ell \leftarrow t \wedge \neg ab_t$ $ab_e \leftarrow \bot$ $ab_t \leftarrow \bot$	$e \leftarrow \top$ $e \leftarrow \bot$ $t \leftarrow \top$ $t \leftarrow \bot$	ℓ	$e \leftarrow \top$	$t \leftarrow \top$	0.13
9	$\ell \leftarrow e \wedge \neg ab_e$ $\ell \leftarrow o \wedge \neg ab_o$ $ab_e \leftarrow \bot$ $ab_o \leftarrow \bot$ $ab_e \leftarrow \neg o$ $ab_o \leftarrow \neg e$	$e \leftarrow \top$ $e \leftarrow \bot$ $o \leftarrow \top$ $o \leftarrow \bot$	ℓ	$e \leftarrow \top$ $o \leftarrow \top$		0.54

The Suppression Task – Experiments 10-12

Ex.	$\mathcal{P}$	$\mathcal{A}_{\mathcal{P}}$	$\mathcal{O}$	$\mathcal{X}$	$\neg e$
10	$\ell \leftarrow e \wedge \neg ab_e$ $ab_e \leftarrow \bot$	$e \leftarrow \top$ $e \leftarrow \bot$	$\neg \ell$	$e \leftarrow \bot$	0.96
11	$\ell \leftarrow e \wedge \neg ab_e$ $\ell \leftarrow t \wedge \neg ab_t$ $ab_e \leftarrow \bot$ $ab_t \leftarrow \bot$	$e \leftarrow \top$ $e \leftarrow \bot$ $t \leftarrow \top$ $t \leftarrow \bot$	$\neg \ell$	$e \leftarrow \bot$ $t \leftarrow \bot$	0.96
12	$\ell \leftarrow e \wedge \neg ab_e$ $\ell \leftarrow o \wedge \neg ab_3$ $ab_e \leftarrow \bot$ $ab_3 \leftarrow \bot$ $ab_e \leftarrow \neg o$ $ab_3 \leftarrow \neg e$	$e \leftarrow \top$ $e \leftarrow \bot$ $o \leftarrow \top$ $o \leftarrow \bot$	$\neg \ell$	$e \leftarrow \bot$	$o \leftarrow \bot$	0.33

Skeptical and Credulous Consequences

- Let $\langle \mathcal{P}, \mathcal{A}_\mathcal{P}, \mathcal{IC}, \models_{wcs} \rangle$ be an abductive framework, $\mathcal{O}$ an observation, and F a formula
- *F follows credulously from $\mathcal{P}$ and $\mathcal{O}$*
  iff there exists an explanation $\mathcal{X}$ for $\mathcal{O}$ such that $\mathcal{P} \cup \mathcal{X} \models_{wcs} F$
- *F follows skeptically from $\mathcal{P}$ and $\mathcal{O}$*
  iff for all explanations $\mathcal{X}$ for $\mathcal{O}$ we find $\mathcal{P} \cup \mathcal{X} \models_{wcs} F$

Complementary Pairs

- A pair of clauses of the form $c \leftarrow \top$ and $c \leftarrow \bot$ is complementary
- A set of clauses is complementary if it contains a complementary pair
- Proposition 29 Let $\langle \mathcal{P}, \mathcal{A}_{\mathcal{P}}, \mathcal{IC}, \models_{wcs} \rangle$ be an abductive framework $\mathcal{O}$ an observation and $\mathcal{X} \subseteq \mathcal{A}_{\mathcal{P}}$ an explanation for $\mathcal{O}$ which contains a complementary pair $c \leftarrow \top$ and $c \leftarrow \bot$
  - Then, $\mathcal{X}' = \mathcal{X} \setminus \{c \leftarrow \bot\}$ is also an explanation for $\mathcal{O}$ and $\mathcal{M}_{wc(\mathcal{P} \cup \mathcal{X})} = \mathcal{M}_{wc(\mathcal{P} \cup \mathcal{X}')}$
- Proof $\rightsquigarrow$ Exercise
- Proposition 30 Given n undefined atoms in a ground program $\mathcal{P}$ there are 2^{2^n} subsets of $\mathcal{A}_{\mathcal{P}}$ and 3^n non-complementary subsets of $\mathcal{A}_{\mathcal{P}}$
- Proof $\rightsquigarrow$ Exercise
- Are humans considering $3^n - 1$ possible explanations?

Reasoning to the Best Explanation 1

- If I watered the garden, then the grass is wet
  If it was raining, then the grass is wet
- **Reasoning towards a program**

$$\begin{array}{rcl} wet\_grass & \leftarrow & watered \wedge \neg ab_{watered} \\ ab_{watered} & \leftarrow & \bot \\ wet\_grass & \leftarrow & rain \wedge \neg ab_{rain} \\ ab_{rain} & \leftarrow & \bot \end{array}$$

- **Observation** The grass is wet
- **What are the minimal explanations?**

Reasoning to the Best Explanation 2

- If I watered the garden, then the grass is wet
  If it was raining, then the grass is wet
  The sky was clear all day

- Reasoning towards a program

$$\begin{array}{rcl} wet\_grass & \leftarrow & watered \wedge \neg ab_{watered} \\ ab_{watered} & \leftarrow & \bot \\ wet\_grass & \leftarrow & rain \wedge \neg ab_{rain} \\ ab_{rain} & \leftarrow & \bot \\ clear\_sky & \leftarrow & \top \end{array}$$

- Common sense $\mathsf{U} \leftarrow clear\_sky \wedge rain$

- Observation The grass is wet

- What is the best minimal explanation?

The Tweety Scenario 1

- **Birds usually fly, but kiwis and penguins do not; Tweety and Jerry are birds**
- **Reasoning towards a program**

$$\begin{array}{rcl} \textit{fly } X & \leftarrow & \textit{bird } X \wedge \neg \textit{ab}_{\textit{fly}}\ X \\ \textit{ab}_{\textit{fly}}\ X & \leftarrow & \textit{kiwi } X \\ \textit{ab}_{\textit{fly}}\ X & \leftarrow & \textit{penguin } X \\ \textit{bird tweety} & \leftarrow & \top \\ \textit{bird jerry} & \leftarrow & \top \end{array}$$

- **The least model of its weak completion**

$$\langle \{\textit{bird tweety},\ \textit{bird jerry}\}, \emptyset \rangle$$

- **The set of abducibles**

$$\begin{array}{rclcrcl} \textit{kiwi tweety} & \leftarrow & \top & \quad & \textit{kiwi tweety} & \leftarrow & \bot \\ \textit{kiwi jerry} & \leftarrow & \top & & \textit{kiwi jerry} & \leftarrow & \bot \\ \textit{penguin tweety} & \leftarrow & \top & & \textit{penguin tweety} & \leftarrow & \bot \\ \textit{penguin jerry} & \leftarrow & \top & & \textit{penguin jerry} & \leftarrow & \bot \end{array}$$

The Tweety Scenario 2

- Birds usually fly, but kiwis and penguins do not; Tweedy and Jerry are birds
- Suppose we observe that *Jerry does fly*
- The minimal explanation is

$$\mathcal{X} \quad = \quad \{\textit{kiwi jerry} \leftarrow \bot,\ \textit{penguin jerry} \leftarrow \bot\},$$

- The observation follows
- Are you happy with this formalization?

The Tweety Scenario 3

- Birds usually fly; Tweety and Jerry are birds
- Reasoning towards a program

$$\begin{array}{rcl} \textit{fly}\ X & \leftarrow & \textit{bird}\ X \wedge \neg \textit{ab}_{\textit{fly}}\ X \\ \textit{ab}_{\textit{fly}}\ X & \leftarrow & \bot \\ \textit{bird tweety} & \leftarrow & \top \\ \textit{bird jerry} & \leftarrow & \top \end{array}$$

- The least model of its weak completion

$$\langle\{\textit{bird tweety},\ \textit{bird jerry},\ \textit{fly tweety},\ \textit{fly jerry}\},\{\textit{ab}_{\textit{fly}}\ \textit{tweety},\ \textit{ab}_{\textit{fly}}\ \textit{jerry}\}\rangle.$$

- What is the set of abducibles in this case?
- Can the observation that Tweety does not fly be explained?
- Are you happy with this formalization?

Summary of Chapter 3

- **Programs as well as their weak completions admit least models under the three-valued Łukasiewicz logic**
  - **This does not hold if Kleene or Fitting logic is used**
- **The least models of weakly completed programs can be computed as least fixed points of an associated semantic operator**
- **These computations are bounded by the first limit ordinal in case of finite propositional programs, finite datalog programs or acyclic programs**
- **Abduction can be applied to explain observations**
  - **Humans seem to apply skeptical abduction**
- **The approach adequately models an average human reasoner in the suppression task**
- **All results hold in the presence of an equational theory**

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