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Human Reasoning and the Weak Completion Semantics

TECHNISCHE UNIVERSITÄT DRESDEN

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This Master is run under the context of Action
No 2020-EU-IA-0087, co-financed by the EU CEF Telecom
under GA nr. INEA/CEF/ICT/A2020/2267423

Human Reasoning and the Weak Completion Semantics

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- **Human Reasoning**
- **Some History**
- **Subareas**
- **Table of Contents**
- **Working Material**

Human Reasoning

- **Instructions on the boarding card distributed at Amsterdam Schiphol Airport**
  - ***If it's thirty minutes before your flight departure, make your way to the gate***
    As soon as the gate number is confirmed, make your way to the gate
- **Notice in the London Underground**
  - ***If there is an emergency, then you press the alarm signal bottom***
    The driver will stop, if any part of the train is in a station
- **Observations**
  - **Intended meaning differs from literal meaning**
  - **Rigid adherence to classical logic is no help in modeling the examples**
  - **There seems to be a reasoning process towards more plausible meanings**
    - ***The driver will stop the train in a station, if the driver is alerted to an emergency and any part of the train is in the station***
      Kowalski: Computational Logic and Human Life 2011

Human Reasoning – More Examples

- What follows from the following sentences?
  - *If I solve all exercises, then I will pass the exam*
    I solve all exercises
  - *If I do not water my plants, then they will die*
    I water my plants
  - *In some cases when I go out, I am not in company*
    Every time I am very happy, I am in company
- What are adequate models of human reasoning?
- Can logics adequately model human reasoning?
- Are models formal, computational, and cognitive?

Some History from a Personal View

- **Logic programming and logic based knowledge representation and reasoning**
  - **Least models**
- **Neural-symbolic integration**
  - **Connectionist model generation**
- **Models versus mental models**
- **Errors in** Stenning, van Lambalgen: Human Reasoning and Cognitive Science 2008
  - Łukasiewicz: O logice trójwartościowey 1920
  - H., Kencana Ramli:
    Logic Programs under Three-Valued Łukasiewicz's Semantics 2009
  - **Weak Completion Semantics (WCS)**
- Khemlani, Johnson-Laird: Theories of Syllogisms: A Meta Analysis 2012
  - **WCS outperformed 12 cognitive theories**

Subareas

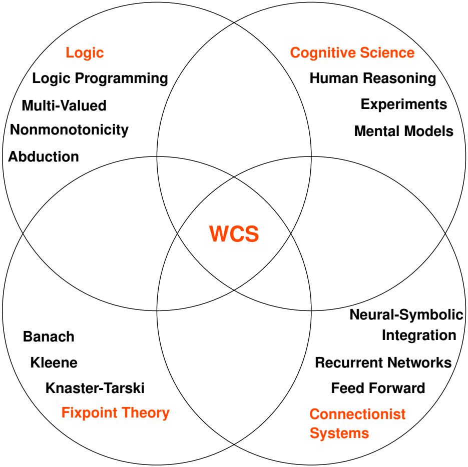
Logic
Logic Programming
Multi-Valued
Nonmonotonicity
Abduction
Cognitive Science
Human Reasoning
Experiments
Mental Models
WCS
Banach
Kleene
Knaster-Tarski
Fixpoint Theory
Neural-Symbolic
Integration
Recurrent Networks
Feed Forward
Connectionist
Systems

Table of Contents

Working Material

- **A manuscript will be available**
- **All references are given in the manuscript**

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The Weak Completion Semantics – Introduction

Steffen Hölldobler
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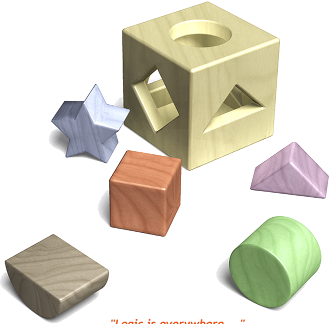


"Logic is everywhere ..."

- **Human Reasoning and Deduction**
- **The Goal**
- **The Suppression Task**
- **Counterexamples**
- **Further Remarks**
- **Suggested Readings**

Human Reasoning and Deduction

- Johnson-Laird, Byrne: Deduction 1991

  ***You need to make deductions to formulate plans and to evaluate actions;
  to determine the consequences of assumptions and hypotheses;
  to interpret and to formulate instructions, rules, and general principles;
  to pursue arguments and negotiations;
  to weigh evidence and to assess data;
  to decide between competing theories;
  and to solve problems.
  A world without deduction would be a world without science, technology, laws, social conventions, and culture.***

- Johnson-Laird: Models of Deduction 1984

  Are there any general ways of thinking that humans follow when they make deductions?

The Goal

- The development of a cognitive theory for adequately modelling human reasoning tasks
  - computational
  - comprehensive
  - a connectionist realization
- Background
  - logic programming
  - logic-based knowledge representation and reasoning

The Suppression Task

- **12 experiments carried out by Ruth Byrne in the 1980s**
- **Repeated several times leading to similar results**
- **Showing that humans suppress previously drawn inferences**
  - **valid inferences**
  - **invalid inferences**
  - **with respect to classical two-valued logic**

  Byrne: Suppressing Valid Inferences with Conditionals 1989

Affirmation of the Antecedent

- *She has an essay to write*
  If she has an essay to write, then she will study late in the library
  Will she study late in the library?
  - **96% yes**
- *She has an essay to write*
  If she has an essay to write, then she will study late in the library
  If she has textbooks to read, then she will study late in the library
  Will she study late in the library?
  - **96% yes**
- *She has an essay to write*
  If she has an essay to write, then she will study late in the library
  If the library stays open, then she will study late in the library
  Will she study late in the library?
  - **38% yes**

Naive Two-Valued Classical Logic

- $\{e,\ e \rightarrow \ell\} \models \ell$
  - ok 96%
  - Modus ponens
- $\{e,\ e \rightarrow \ell,\ t \rightarrow \ell\} \models \ell$
  - ok 96%
  - Two-valued classical logic is monotonic
- $\{e,\ e \rightarrow \ell,\ o \rightarrow \ell\} \models \ell$
  - Upps only 38% of the participants were doing this
  - Human reasoning appears to be nonmonotonic

Adequateness

- **Two-valued classical logic is universal**
  - **If human reasoning can be computed, then we should be able to model the three experiments in two-valued classical logic** **How?**
- Bibel: Perspectives on Automated Deduction 1991

  There is an adequate general proof method that can automatically discover any proof done by humans provided the problem (including all required knowledge) is stated in appropriately formalized terms

  Adequateness is understood as the property of a theorem proving method that

  for any given knowledge base, the method solves simpler problems faster than more difficult ones

  Simplicity is measured under consideration of all (general) formalisms available to capture the problem and intrinsic in this assumption is a belief in the existence of an algorithm that is feasible (from a complexity point of view) for the set of problems humans can solve

Towards a Simple Formalism to Capture the Suppression Task

- **We need to answer the following questions**
  - **If the participants in the third experiment did not use two-valued classical logic what else did they use?**
  - **How did they come up with their answers?**
  - **Can we formally specify a system in which the three experiments can be uniformly modeled such that the answers given by the majority of the participants can be computed?**
- **My proposal**
  - **Take a nonmonotonic and multi-valued logic**
  - **Take the Weak Completion Semantics**

The Weak Completion Semantics in a Nutshell

- **Inspired by**
  Stenning, van Lambalgen: Human Reasoning and Cognitive Science 2008
- **The six stages of reasoning according to the Weak Completion Semantics**
  - **Reasoning towards a (logic) program**
  - **Weakly completing the program**
  - **Computing its least model**
  - **Reasoning with respect to the least model**
  - **If necessary, applying skeptical abduction**
  - **If possible, searching for counterexamples**

Affirmation of the Antecedent

- ***She has an essay to write***
  If she has an essay to write, then she will go to the library

- **Program $\mathcal{P}$**

$e \leftarrow \top$	fact	definition of e
$\ell \leftarrow e \wedge \neg ab_e$	rule	definition of ℓ
$ab_e \leftarrow \bot$	assumption	ab_e is assumed to be false

- **Weakly completed program & Generation of least model**

$$e \leftrightarrow \top$$
$$\ell \leftrightarrow e \wedge \neg ab_e$$
$$ab_e \leftrightarrow \bot$$

true	*false*	$\Phi_{\mathcal{P}}$
e	ab_e	1
ℓ		2

- **Computing logical consequences with respect to the least model**
  - **She will go to the library**

Łukasiewicz Three-Valued Logic

- Łukasiewicz: O logice trójwartościowey 1920

F	$\neg F$
$\top$	$\bot$
$\bot$	$\top$
U	U

$\wedge$	$\top$	U	$\bot$
$\top$	$\top$	U	$\bot$
U	U	U	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$

$\vee$	$\top$	U	$\bot$
$\top$	$\top$	$\top$	$\top$
U	$\top$	U	U
$\bot$	$\top$	U	$\bot$

$\leftarrow$	$\top$	U	$\bot$
$\top$	$\top$	$\top$	$\top$
U	U	$\top$	$\top$
$\bot$	$\bot$	U	$\top$

$\leftrightarrow$	$\top$	U	$\bot$
$\top$	$\top$	U	$\bot$
U	U	$\top$	U
$\bot$	$\bot$	U	$\top$

Affirmation of the Antecedent and Alternative Arguments

- *She has an essay to write*
  If she has an essay to write, then she will go to the library
  If she has textbooks to read, then she will go to the library

- Program $\mathcal{P}$

$e \leftarrow \top$	fact	definition of e
$\ell \leftarrow e \wedge \neg ab_e$	rule	definition of ℓ
$ab_e \leftarrow \bot$	assumption	ab_e is assumed to be false
$\ell \leftarrow t \wedge \neg ab_t$	rule	definition of ℓ
$ab_t \leftarrow \bot$	assumption	ab_t is assumed to be false

- Weakly completed program & Generation of least model

$$
\begin{aligned}
e &\leftrightarrow \top \\
\ell &\leftrightarrow (e \wedge \neg ab_e) \vee (t \wedge \neg ab_t) \\
ab_e &\leftrightarrow \bot \\
ab_t &\leftrightarrow \bot
\end{aligned}
$$

true	false	$\Phi_{\mathcal{P}}$
e	ab_e	1
	ab_t	
ℓ		2

- Computing logical consequences with respect to the least model
  - She will go to the library

Reasoning Towards an Appropriate Logical Form

- *If she has an essay to write, then she will go to the library*
  If the library stays open, then she will go to the library
- Kowalski: Computational Logic and Human Thinking 2011
- **Context independent rules**
  - *If she has an essay to write and the library stays open,*
    then she will study late in the library
    If the library stays open and she has a reason for studying in the library,
    then she will study late in the library
- **Context dependent rule plus exception**
  - *If she has an essay to write, then she will study late in the library*
    However, if the library does not stay open, then she will not study late in the library
  - **The last statement is the contrapositive of the converse of the original sentence!**

Affirmation of the Antecedent and Additional Arguments

- *She has an essay to write*
  If she has an essay to write, then she will go to the library
  If the library stays open, then she will go to the library

- **Programs $\mathcal{P}$**

$e \leftarrow \top$	fact	definition of e
$\ell \leftarrow e \wedge \neg ab_e$	rule	definition of ℓ
$ab_e \leftarrow \bot$	assumption	ab_e is assumed to be false
$\ell \leftarrow o \wedge \neg ab_o$	rule	definition of ℓ
$ab_o \leftarrow \bot$	assumption	ab_o is assumed to be false
$ab_e \leftarrow \neg o$	rule	definition of ab_e
$ab_o \leftarrow \neg e$	rule	definition of ab_o

- **Weakly completed program & Generation of least model**

$$
\begin{array}{rcl}
e & \leftrightarrow & \top \\
\ell & \leftrightarrow & (e \wedge \neg ab_e) \vee (o \wedge \neg ab_o) \\
ab_e & \leftrightarrow & \bot \vee \neg o \\
ab_o & \leftrightarrow & \bot \vee \neg e
\end{array}
$$

true	*false*	$\Phi_{\mathcal{P}}$
e		1
	ab_o	2

- **Computing logical consequences with respect to the least model**
  - We can neither conclude that she will go nor that she will not go to the library

Denial of the Antecedent

- *She does not have an essay to write*
  If she has an essay to write, then she will study late in the library
  Will she not study late in the library?
  - 46% yes
- *She does not have an essay to write*
  If she has an essay to write, then she will study late in the library
  If she has textbooks to read, then she will study late in the library
  Will she not study late in the library?
  - 4% yes
- *She does not have an essay to write*
  If she has an essay to write, then she will study late in the library
  If the library stays open, then she will study late in the library
  Will she not study late in the library?
  - 63% yes

Denial of the Antecedent

- *She does not have an essay to write*
  If she has an essay to write, then she will go to the library
- Program $\mathcal{P}$

$$
\begin{array}{rcll}
e & \leftarrow & \bot & \text{assumption} \\
\ell & \leftarrow & e \wedge \neg ab_e & \text{rule} \\
ab_e & \leftarrow & \bot & \text{assumption}
\end{array}
$$

- Weakly completed program & Generation of least model

$$
\begin{array}{rcl}
e & \leftrightarrow & \bot \\
\ell & \leftrightarrow & e \wedge \neg ab_e \\
ab_e & \leftrightarrow & \bot
\end{array}
$$

true	*false*	$\Phi_{\mathcal{P}}$
	e	1
	ab_e	
	ℓ	2

- Computing logical consequences with respect to the least model
  - She will not go to the library

Denial of the Antecedent and Alternative Arguments

- ***She does not have an essay to write***
  If she has an essay to write, then she will go to the library
  If she has textbooks to read, then she will go to the library

- **Program $\mathcal{P}$**

$$
\begin{array}{rcll}
e & \leftarrow & \bot & \text{assumption} \\
\ell & \leftarrow & e \wedge \neg ab_e & \text{rule} \\
ab_e & \leftarrow & \bot & \text{assumption} \\
\ell & \leftarrow & t \wedge \neg ab_t & \text{rule} \\
ab_t & \leftarrow & \bot & \text{assumption}
\end{array}
$$

- **Weakly completed program & Generation of least model**

$$
\begin{array}{rcl}
e & \leftrightarrow & \bot \\
\ell & \leftrightarrow & (e \wedge \neg ab_e) \vee (t \wedge \neg ab_t) \\
ab_e & \leftrightarrow & \bot \\
ab_t & \leftrightarrow & \bot
\end{array}
$$

true	false	$\Phi_{\mathcal{P}}$
	e	1
	ab_e	
	ab_t	

- **Computing logical consequences with respect to the least model**
  - We can neither conclude that she will go nor that she will not go to the library

Denial of the Antecedent and Additional Arguments

- *She does not have an essay to write*
  If she has an essay to write, then she will go to the library
  If the library stays open, then she will go to the library

- Programs $\mathcal{P}$

$$
\begin{array}{rcll}
e & \leftarrow & \bot & \text{assumption} \\
\ell & \leftarrow & e \wedge \neg ab_e & \text{rule} \\
ab_e & \leftarrow & \bot & \text{assumption} \\
\ell & \leftarrow & o \wedge \neg ab_o & \text{rule} \\
ab_o & \leftarrow & \bot & \text{assumption} \\
ab_e & \leftarrow & \neg o & \text{rule} \\
ab_o & \leftarrow & \neg e & \text{rule}
\end{array}
$$

- Weakly completed program & Generation of least model

$$
\begin{array}{rcl}
e & \leftrightarrow & \bot \\
\ell & \leftrightarrow & (e \wedge \neg ab_e) \vee (o \wedge \neg ab_o) \\
ab_e & \leftrightarrow & \bot \vee \neg o \\
ab_o & \leftrightarrow & \bot \vee \neg e
\end{array}
$$

true	*false*	$\Phi_{\mathcal{P}}$
	e	1
ab_o		2
	ℓ	3

- Computing logical consequences with respect to the least model
  - **She will not go to the library**

Affirmation of the Consequent

- *She will study late in the library*
  If she has an essay to write, then she will study late in the library
  Has she an essay to write?
  - 71% yes
- *She will study late in the library*
  If she has an essay to write, then she will study late in the library
  If she has textbooks to read, then she will study late in the library
  Has she an essay to write?
  - 13% yes
- *She will study late in the library*
  If she has an essay to write, then she will study late in the library
  If the library stays open, then she will study late in the library
  Has she an essay to write?
  - 54% yes

Affirmation of the Consequent

- *She will go to the library*
  If she has an essay to write, then she will go to the library

- **Program**

$$\begin{array}{rcl} \ell & \leftarrow & \top \\ \ell & \leftarrow & e \wedge \neg ab_e \\ ab_e & \leftarrow & \bot \end{array}$$

- **Weakly completed program & Generation of least model**

$$\begin{array}{rcl} \ell & \leftrightarrow & \top \vee (e \wedge \neg ab_e) \\ ab_e & \leftrightarrow & \bot \end{array}$$

true	*false*
ℓ	ab_e

- **Computing logical consequences with respect to the least model**
  - **We cannot conclude that she has an essay to write**
  - **But most humans conclude that she has**
  - **Don't consider ℓ as a fact in the presence of a rule for ℓ**
    - **Consider ℓ to be an observation that needs to be explained**

Abduction

- **Program & Observation**

$$\begin{array}{rcl} \ell & \leftarrow & e \wedge \neg ab_e \\ ab_e & \leftarrow & \bot \end{array}$$

ℓ

- **Abducibles**

$$e \leftarrow \top \qquad e \leftarrow \bot$$

- **Weakly completed program plus explanation & Generation of least model**

$$\begin{array}{rcl} \ell & \leftrightarrow & e \wedge \neg ab_e \\ ab_e & \leftrightarrow & \bot \\ e & \leftrightarrow & \top \end{array}$$

true	false
e	ab_e
ℓ	

- **Computing logical consequences with respect to the least model**
  - She has an essay to write

Affirmation of the Consequent and Alternative Arguments

- **Program & Observation**

$$\begin{array}{rcl} \ell & \leftarrow & e \wedge \neg ab_e \\ ab_e & \leftarrow & \bot \\ \ell & \leftarrow & t \wedge \neg ab_t \\ ab_t & \leftarrow & \bot \end{array} \qquad \ell$$

- **Abducibles**

$$e \leftarrow \top \qquad t \leftarrow \top \qquad e \leftarrow \bot \qquad t \leftarrow \bot$$

- **Weakly completed program plus explanations & Generation of least models**

$$\begin{array}{rcl} \ell & \leftrightarrow & (e \wedge \neg ab_e) \vee (t \wedge \neg ab_t) \\ ab_e & \leftrightarrow & \bot \\ ab_t & \leftrightarrow & \bot \\ e & \leftrightarrow & \top \quad \text{or} \quad t \leftrightarrow \top \end{array}$$

true	false
e	ab_e
	ab_t
ℓ	

true	false
t	ab_e
	ab_t
ℓ	

- **Computing skeptical consequences with respect to both models**
  - We cannot conclude that she has an essay to write
  - Reasoning creduluously we can but the participants did not do this

Affirmation of the Consequent and Additional Arguments

- **Program & Observation**

$$\begin{array}{rcl} \ell & \leftarrow & e \wedge \neg ab_e \\ ab_e & \leftarrow & \bot \\ \ell & \leftarrow & o \wedge \neg ab_o \\ ab_o & \leftarrow & \bot \\ ab_e & \leftarrow & \neg o \\ ab_o & \leftarrow & \neg e \end{array}$$

$$\ell$$

- **Abducibles**

$$e \leftarrow \top \qquad o \leftarrow \top \qquad e \leftarrow \bot \qquad o \leftarrow \bot$$

- **Weakly completed program plus explanations & Generation of least model**

$$\begin{array}{rcl} \ell & \leftrightarrow & (e \wedge \neg ab_e) \vee (o \wedge \neg ab_o) \\ ab_e & \leftrightarrow & \bot \vee \neg o \\ ab_o & \leftrightarrow & \bot \vee \neg e \\ e & \leftrightarrow & \top \\ o & \leftrightarrow & \top \end{array}$$

true	*false*
e	
o	
	ab_e
	ab_o
ℓ	

- **Computing consequences with respect to the least model**
  - **She has an essay to write**

Denial of the Consequent

- *She will not study late in the library*
  If she has an essay to write, then she will study late in the library
  Does she not have an essay to write?
  - **92% yes**
- *She will not study late in the library*
  If she has an essay to write, then she will study late in the library
  If she has textbooks to read, then she will study late in the library
  Does she not have essay to write?
  - **96% yes**
- *She will not study late in the library*
  If she has an essay to write, then she will study late in the library
  If the library stays open then, she will study late in the library
  Does she not have an essay to write?
  - **33% yes**

Denial of the Consequent

- **Program & Observation**

$$\ell \leftarrow e \wedge \neg ab_e \qquad\qquad \neg\ell$$
$$ab_e \leftarrow \bot$$

- **Abducibles**

$$e \leftarrow \top \qquad\qquad e \leftarrow \bot$$

- **Weakly completed program plus explanation & Generation of least model**

$$\ell \leftrightarrow e \wedge \neg ab_e$$
$$ab_e \leftrightarrow \bot$$
$$e \leftrightarrow \bot$$

true	false
	ab_e
	e
	ℓ

- **Computing logical consequences with respect to the least model**
  - She does not have an essay to write

Denial of the Consequent and Alternative Arguments

- **Program & Observation**

$$\begin{array}{rcl} \ell & \leftarrow & e \wedge \neg ab_e \\ ab_e & \leftarrow & \bot \\ \ell & \leftarrow & t \wedge \neg ab_t \\ ab_t & \leftarrow & \bot \end{array} \qquad \neg \ell$$

- **Abducibles**

$$e \leftarrow \top \qquad t \leftarrow \top \qquad e \leftarrow \bot \qquad t \leftarrow \bot$$

- **Weakly completed program plus explanations & Generation of least model**

$$\begin{array}{rcl} \ell & \leftrightarrow & (e \wedge \neg ab_e) \vee (t \wedge \neg ab_t) \\ ab_e & \leftrightarrow & \bot \\ ab_t & \leftrightarrow & \bot \\ e & \leftrightarrow & \bot \\ t & \leftrightarrow & \bot \end{array}$$

true	false
	e
	t
	ab_e
	ab_t
	ℓ

- **Computing consequences with respect to the least model**
  - She does not have an essay to write

Denial of the Consequent and Additional Arguments

- **Program & Observation**

$$\begin{array}{rcl} \ell & \leftarrow & e \wedge \neg ab_e \\ ab_e & \leftarrow & \bot \\ \ell & \leftarrow & o \wedge \neg ab_o \\ ab_o & \leftarrow & \bot \\ ab_e & \leftarrow & \neg o \\ ab_o & \leftarrow & \neg e \end{array} \qquad \neg \ell$$

- **Abducibles**

$$e \leftarrow \top \qquad o \leftarrow \top \qquad e \leftarrow \bot \qquad o \leftarrow \bot$$

- **Weakly completed program plus explanations & Generation of least models**

$$\begin{array}{rcl} \ell & \leftrightarrow & (e \wedge \neg ab_e) \vee (o \wedge \neg ab_o) \\ ab_e & \leftrightarrow & \bot \vee \neg o \\ ab_o & \leftrightarrow & \bot \vee \neg e \\ e & \leftrightarrow & \bot \quad \text{or} \quad o \leftrightarrow \bot \end{array}$$

true	*false*
	e
ab_o	
	ℓ

true	*false*
	o
ab_e	
	ℓ

- **Computing skeptical consequences with respect to both models**
  - **We cannot conclude that she does not have an essay to write**

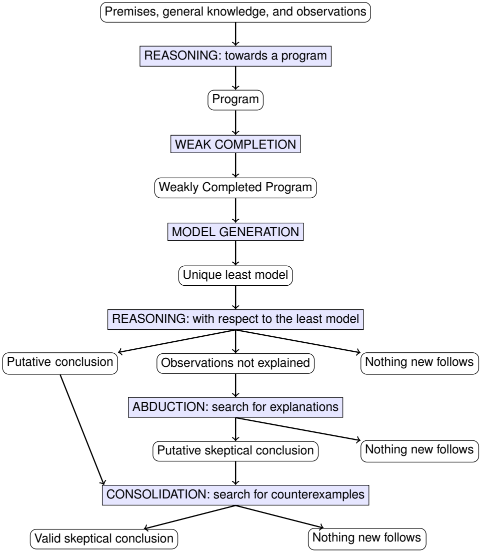
Summary (1)

Ex	atomic sentences				conditional sentences			queries				WCS
	e	$\neg e$	ℓ	$\neg\ell$	$e \Rightarrow \ell$	$t \Rightarrow \ell$	$o \Rightarrow \ell$	ℓ	$\neg\ell$	e	$\neg e$	
1	X				X			96%				$\top$
2	X				X	X		96%				$\top$
3	X				X		X	38%				**U**
4		X			X				46%			$\top$
5		X			X	X			4%			**U**
6		X			X		X		63%			$\top$
7			X		X					71%		$\top$
8			X		X	X				13%		**U**
9			X		X		X			54%		$\top$
10				X	X						92%	$\top$
11				X	X	X					96%	$\top$
12				X	X		X				33%	**U**

Summary (2)

- The Weak Completion Semantics appears to be adequate
  - The suppression effect is modeled
  - The average reasoner is modeled
- Principles
  - Licenses for inference
    - Abnormalities
    - Modeling additional antecedents by context dependent rules
  - Abduction
    - If a fact corresponds to the consequent of a conditional
      then treat the fact as an observation which needs to be explained
    - Skeptical abduction is adequate
    - Credulous abduction is not

The Six Stages of Reasoning



Necessary and Non-Necessary Antecedents

- Given a conditional sentence *if A then C*
  - *A* is necessary iff *C* cannot be true unless *A* is true
  - *A* is non-necessary iff *C* can be true irrespective of the truth of *A*
- Are the following antecedents necessary or non-necessary?
  - *If the library stays open, then she will study late in the library*
  - *If she has an essay to write, then she will study late in the library*
- The answer depends on experience, culture, etc

Non-Necessary Antecedents

- Suppose the antecedent of

  if she has an essay to write, then she will study late in the library

  was classified as non-necessary
- But then there are other (unknown) reasons for studying late in the library
  - This can be taken into consideration by the abducible $\ell \leftarrow \top$
- Recall Experiment 4 (denial of the antecedent)
  - The least model was $\langle \emptyset, \{e, \ell, ab_e\} \rangle$
  - 46% answered *she will not study late in the library*
  - What about the others?
  - Due to the abducible we can construct a counterexample $\langle \{\ell\}, \{e, ab_e\} \rangle$
  - Reasoning skeptically *she may or may not study late in the library*

Formal and/or Cognitive Theory

- Collins English Dictionary
  - **A formal theory is an uninterpreted symbolic system whose syntax is precisely defined and on which a relation of deducibility is defined in purely syntactic terms**
  - **A cognitive theory is any theory of mind that focuses on mental activities, such as perceiving, attending, thinking, remembering, evaluating, planning, language, and creativity, especially a theory that suggests a model for the various processes involved**
- **The Weak Completion Semantics is a formal theory**
- **But is it also a cognitive theory?**

Human Disjunctive Reasoning

- **In classical two-valued logic $\{A \vee B, \neg A\} \models B$ holds**
  - **Can you prove it?**
- **What do you think about the following human reasoning episode?**

  Eva's in Rio or she's in Brazil
  She's not in Brazil
  Therefore, she is in Rio

  - Johnson-Laird, Byrne: Conditionals 2002
    No sensible person other than a logician is likely to draw this conclusion as it is impossible for Eva to be in Rio and not in Brazil, because Rio is in Brazil
  - **What should a computer scientist reply?**

Expected and Suggested Readings

- **I expect students to read**
  - Byrne: Suppressing Valid Inferences with Conditionals 1989
  - Łuksiewicz: O logice trójwartościowey 1920
- **I suggest that students have a look at**
  - Stenning, van Lambalgen: Human Reasoning and Cognitive Science 2008
  - Kowalski: Computational Logic and Human Thinking 2011
- **Complete references are given in the manuscript**

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Human Reasoning and the Weak Completion Semantics

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Foundations – Logics

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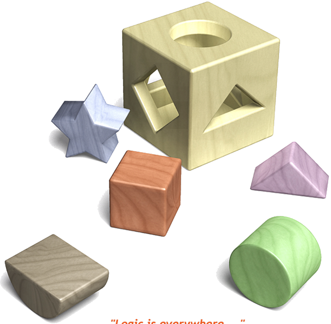
- **Alphabet**
- **Formulas**
- **Equations**
- **Interpretations**
- **Models**
- **Semantic Equivalence**
- **Logical Consequence**



"Logic is everywhere ..."

Alphabet

- We consider an alphabet consisting of
  - a finite set of function symbols with arity ≥ 0
  - a countably infinite set of variables
  - a finite or countably infinite set of relation symbols with arity ≥ 0
  - the connectives $\neg$, $\wedge$, $\vee$, $\leftarrow$, and $\leftrightarrow$
  - the existential quantifier $\exists$
  - the universal quantifier $\forall$
  - the special symbols (,), $\top$, $\bot$, U, and ,
- We assume that alphabets are implicitly given as the set of symbols occurring in the syntactic objects under consideration

Terms, Atoms, and Literals

- The set of terms is the smallest set satisfying the following conditions:
  - Each variable is a term
  - If f is an n-ary function symbol, $n \geq 0$, and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a term as well
    - $X, Y, a, b, f(a, X), g\,a$
- The set of atoms consists of all expressions of the form $p(t_1, \ldots, t_n)$, where p is an n-ary relation symbol, $n \geq 0$, and $t_1, \ldots, t_n$ are terms
  - $p(b, X), q, r, s\,a$
- A literal is either an atom or its negation
  - $p(a, b), \neg p(a, b)$
- A term, atom, or literal is said to be ground iff if it does not contain the occurrence of a variable

Formulas

- The set of formulas is the smallest set satisfying the following conditions
  - Each atom and each truth constant is a formula
  - If F is a formula, then so is $\neg F$
  - If F and G are formulas, then so are $(F \wedge G)$, $(F \vee G)$, $(F \leftarrow G)$, and $(F \leftrightarrow G)$
  - If F is a formula and X is a variable, then $(\forall X)\ F$ and $(\exists X)\ F$ are formulas
- A formula is called clause iff if it is a finite disjunction of literals
- In logic programming, clauses are often written in the form

$$A \leftarrow L_1 \wedge \ldots \wedge L_n$$

where A is an atom and L_i, $1 \leq i \leq n$, are literals

  - They are called program clauses with head A and body $L_1 \wedge \ldots \wedge L_n$

Equations

- An equation is an atom of the form $s \approx t$, where
  - s and t are terms and
  - $\approx$ is a binary relation symbol written infix
- Equations are assumed to be universally closed
- We usually consider sets of equations
- Examples
  - $\{a \approx b\}$
  - $\{X \circ 1 \approx X,\ X \circ Y \approx Y \circ X,\ (X \circ Y) \circ Z \approx X \circ (Y \circ Z)\}$ **AC1**

Axioms of Equality

- The equality relation enjoys some typical properties
- They are specified in the following logic program

$$
\begin{array}{rcll}
X \approx X & \leftarrow & \top & \text{reflexivity} \\
X \approx Y & \leftarrow & Y \approx X & \text{symmetry} \\
X \approx Z & \leftarrow & X \approx Y \land Y \approx Z & \text{transitivitiy} \\
f(X_1, \ldots, X_n) \approx f(Y_1, \ldots, Y_n) & \leftarrow & \bigwedge_{i=1}^{n} X_i \approx Y_i & \text{f-substitutivity} \\
r(Y_1, \ldots, Y_n) & \leftarrow & r(X_1, \ldots, X_n) \land \bigwedge_{i=1}^{n} X_i \approx Y_i & \text{r-substitutivity}
\end{array}
$$

where substitutivity axioms are given for each function and relation symbol

Equational Theories

- An equational theory consists of a set ot equations and the axioms of equality
- It is specified by the set of equations
- Example

$$\{X \circ 1 \approx X,\ X \circ Y \approx Y \circ X,\ (X \circ Y) \circ Z \approx X \circ (Y \circ Z)\}$$

specifies the AC1 theory

Finest Congruence Relation

- An equational theory defines a finest congruence relation $\cong$ on the set of ground terms
  - An equational theory is a definite logic program
  - Definite logic programs enjoy the model intersection property
  - The least model is the finest congruence relation
- Let t be a ground term
  - $[t]$ denotes the congruence class defined by $\cong$ and containing t
  - If the set of equations is empty we write t instead of $[t]$

An Abbreviation

- $[p(t_1, \ldots, t_n)]$ is an abbreviation for $p([t_1], \ldots, [t_n])$
- $[p(t_1, \ldots, t_n)] = [q(s_1, \ldots, s_m)]$ iff
  - $p = q$
  - $n = m$ and
  - $[t_i] = [s_i]$ for all $1 \leq i \leq n$
- If the set of equations is empty we write $p(t_1, \ldots, t_n)$ instead of $[p(t_1, \ldots, t_n)]$
- **Example** Consider the AC1-theory

$$
\begin{array}{rcl}
[d \circ t_2] & = & [t_2 \circ d] \\
[d \circ t_1 \circ d] & = & [t_1 \circ d \circ d \circ 1] \\
[p(d \circ t_2, d \circ t_1 \circ d)] & = & [p(t_2 \circ d, t_1 \circ d \circ d \circ 1)]
\end{array}
$$

where d, t_2, t_1, 1 are constants, $\circ$ is a function, and p is a relation symbol

Interpretations and Models

- The **Herbrand universe** is the quotient of the set of ground terms modulo $\cong$
- The **Herbrand base** is the of all expressions of the form $[p(t_1, \ldots, t_n)]$ where
  - p is an n-ary relation symbol and
  - $[t_i]$ are elements of the Herbrand universe for all $1 \leq i \leq n$
- An **interpretation** is a mapping from the set of formulas into the set of truth values such that
  - truth constants are mapped onto themselves and
  - a given equational theory is mapped to true
- An interpretation is defined by
  - the truth tables for the connectives and
  - the mapping of the Herbrand base to the truth values
- An interpretation I is a **model** for a formula F ($I \models F$) iff I maps F to true

Interpretations and Models – Example

- Consider $\mathcal{P} = \{q\,X \leftarrow \neg\, p\,X,\ p\,a \leftarrow \top\}$ and $\mathcal{E} = \{a \approx b\}$
- The Herbrand universe is $\{[a]\}$
- The Herbrand base is $\{[p\,a],\ [q\,b]\}$
- Interpretations are given by the truth tables for the connectives and
  - $[p\,a] \mapsto \top \quad [q\,b] \mapsto \top$
  - $[p\,a] \mapsto \bot \quad [q\,b] \mapsto \top$
  - $[p\,a] \mapsto \top \quad [q\,b] \mapsto \bot$
  - $[p\,a] \mapsto \bot \quad [q\,b] \mapsto \bot$
- Which interpretations are models for $\mathcal{P}$ and $\mathcal{E}$ in classical two-valued logic?

Semantic Equivalence

- Two formulas F and G are **semantically equivalent** ($F \equiv G$)
  iff for all interpretations I we find $I\,F = I\,G$

  - Under which logics is $\bot \vee F \equiv F$ and $(F \leftarrow G) \wedge (G \leftarrow F) \equiv (F \leftrightarrow G)$?

    - Two-valued classical logic
    - Three-valued Łukasiewicz logic
    - Three-valued Kleene logics
    - Three-valued Fitting logic

    Prove your claim

Logical Consequence

- Let $\mathcal{F}$ be a set of formulas and G a formula
- $\mathcal{F}$ logically entails G or G is a logical consequence of $\mathcal{F}$ ($\mathcal{F} \models G$)
  iff every model for $\mathcal{F}$ is also a model for G
- Consider two-valued classical logic
  - Does $\{\ell \leftarrow e,\ e\} \models \ell$ hold?
  - Does $\{\ell \leftarrow e,\ \neg e\} \models \neg\ell$ hold?
  - Does $\{\ell \leftarrow e,\ \ell\} \models e$ hold?
  - Does $\{\ell \leftarrow e,\ \neg\ell\} \models \neg e$ hold?

  Prove your claim

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Human Reasoning and the Weak Completion Semantics

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The Weak Completion Semantics – Theory

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- Programs
- Weakly Completed Programs
- The Meaning of Programs
- Computing Least Models
- Semantic Operators as Contraction Mappings
- Abduction

Programs

- A (normal logic) program $\mathcal{P}$ is a finite or countably infinite set of clauses of the form

$$A \leftarrow Body$$

  - A is an atom (but not an equation) called head
  - $Body$ is either a non-empty conjunction of literals, or $\top$, or $\bot$
- Clauses are assumed to be universally closed
- $A \leftarrow \top$ is called (positive) fact
- $A \leftarrow \bot$ is called (negative) assumption
- All other clauses are called rules
- $\mathcal{P}$ is propositional iff all atoms occurring in $\mathcal{P}$ are propositional
- $\mathcal{P}$ is a datalog program iff the terms occurring in $\mathcal{P}$ are variables and constants
- $\mathcal{P}$ is a definite program iff it does not contain an occurrence of $\bot$ or $\neg$

Programs – Example

- Let $\mathcal{P}$ be

$$\begin{array}{rcl} \ell & \leftarrow & e \wedge \neg ab_e \\ \ell & \leftarrow & t \wedge \neg ab_t \\ e & \leftarrow & \top \\ ab_e & \leftarrow & \bot \\ ab_t & \leftarrow & \bot \end{array}$$

Alphabet

- Let $\mathcal{P}$ be a program and $\mathcal{E}$ an equational theory
  - The alphabet consists precisely of the symbols occurring in $\mathcal{P}$ and $\mathcal{E}$
  - If $\mathcal{P}$ or $\mathcal{E}$ is a first-order program
    then the alphabet must contain at least one constant symbol

Ground Instances

- A **ground instance of a clause** is obtained by replacing each variable occurring in the clause by a ground term
  - The replacement must be consistent in that multiple occurrences of the same variable are replaced by the same ground term
- The **ground instance of a program** $\mathcal{P}$ is the set of all ground instances of clauses occurring in $\mathcal{P}$
  - $g\mathcal{P}$ denotes the ground instance of $\mathcal{P}$
  - If $\mathcal{P}$ is a propositional program then $g\mathcal{P} = \mathcal{P}$

Ground Programs – Example

- Let $\mathcal{P}$ be

$$\begin{array}{rcl} q\,a & \leftarrow & \top \\ q\,s\,X & \leftarrow & q\,X \end{array}$$

- Then $g\mathcal{P}$ consists of

$$\begin{array}{rcl} q\,a & \leftarrow & \top \\ q\,s\,a & \leftarrow & q\,a \\ q\,s\,s\,a & \leftarrow & q\,s\,a \\ \vdots & \vdots & \vdots \end{array}$$

- Is $q\,s\,a \leftarrow q\,s\,a \in g\mathcal{P}$?

Defined Ground Atoms

- Let $\mathcal{P}$ be a ground program, $\mathcal{E}$ an equational theory, and A a ground atom
  - If $\mathcal{E}$ is empty, then A is defined in $\mathcal{P}$ iff $\mathcal{P}$ contains a clause with head A
  - If $\mathcal{E}$ is not empty, then A is defined in $\mathcal{P}$ iff $\mathcal{P}$ contains a clause with head A' and $[A] = [A']$
- A is undefined in $\mathcal{P}$ iff A is not defined in $\mathcal{P}$
- *def* $\mathcal{P}$ denotes the set of defined atoms in $\mathcal{P}$

Defined Ground Atoms – Examples

- **Consider the following programs**

$$\mathcal{E} = \emptyset \qquad\qquad \mathcal{E} = \{a \approx b\}$$

$$\begin{array}{rcl} \ell & \leftarrow & e \wedge \neg ab_e \\ \ell & \leftarrow & t \wedge \neg ab_t \\ e & \leftarrow & \top \\ ab_e & \leftarrow & \bot \\ ab_t & \leftarrow & \bot \end{array} \qquad\qquad \begin{array}{rcl} p\,a & \leftarrow & \top \\ q\,c & \leftarrow & \bot \end{array}$$

  - **How does *def* $\mathcal{P}$ look like?**
  - **Are there any undefined atoms?**

Definitions

- Let $\mathcal{P}$ be a ground program, $\mathcal{E}$ an equational theory, and $\mathcal{S}$ a consistent set of literals
  - If $\mathcal{E}$ is empty, then $defs(\mathcal{P}, \mathcal{S}) = \{A \leftarrow Body \in \mathcal{P} \mid A \in \mathcal{S} \text{ or } \neg A \in \mathcal{S}\}$
  - If $\mathcal{E}$ is not empty, then $defs(\mathcal{P}, \mathcal{S}) = \{A' \leftarrow Body \in \mathcal{P} \mid A \in \mathcal{S} \text{ or } \neg A \in \mathcal{S} \text{ and } [A'] = [A]\}$
- Let $\mathcal{P}$ be

$$
\begin{array}{rcl}
\ell & \leftarrow & e \wedge \neg ab_e \\
\ell & \leftarrow & o \wedge \neg ab_o \\
e & \leftarrow & \top \\
ab_e & \leftarrow & \bot \\
ab_o & \leftarrow & \bot \\
ab_e & \leftarrow & \neg o \\
ab_o & \leftarrow & \neg e
\end{array}
$$

  - How does $defs(\mathcal{P}, \{e, \neg ab_e\})$ look like?

Assumptions

- Let $\mathcal{P}$ be a ground program, $\mathcal{E}$ an equational theory, and A a ground atom
  - If $\mathcal{E}$ is empty then $\neg A$ is assumed in $\mathcal{P}$ iff
    - $\mathcal{P}$ contains an assumption with head A and
    - $\mathcal{P}$ does neither contain a fact $A \leftarrow \top$ nor a rule $A \leftarrow Body$
  - If $\mathcal{E}$ is not empty then $\neg A$ is assumed in $\mathcal{P}$ iff
    - $\mathcal{P}$ contains an assumption of the form $A' \leftarrow \bot$ with $[A] = [A']$ and
    - $\mathcal{P}$ does neither contain a fact $B \leftarrow \top$ nor a rule $B \leftarrow Body$ with $[A] = [B]$
- Why has the second condition been added?

Assumptions – Examples

- What is assumed in the following programs if $\mathcal{E} = \emptyset$?
  - $$\begin{array}{rcl} \ell & \leftarrow & e \wedge \neg ab_e \\ \ell & \leftarrow & t \wedge \neg ab_t \\ e & \leftarrow & \top \\ ab_e & \leftarrow & \bot \\ ab_t & \leftarrow & \bot \end{array}$$
  - $$\begin{array}{rcl} \ell & \leftarrow & e \wedge \neg ab_e \\ \ell & \leftarrow & o \wedge \neg ab_o \\ e & \leftarrow & \top \\ ab_e & \leftarrow & \bot \\ ab_o & \leftarrow & \bot \\ ab_e & \leftarrow & \neg o \\ ab_o & \leftarrow & \neg e \end{array}$$
- Assumptions can be overridden

Weakly Completed Programs

- Let $\mathcal{P}$ be a ground program and $\mathcal{E}$ an equational theory
- Consider the following transformation

1 For all $A \in \mathit{def}\,\mathcal{P}$ do

- If $\mathcal{E}$ is empty, replace all clauses of the form $A \leftarrow Body_1,\ A \leftarrow Body_2,\ \ldots$ occurring in $\mathcal{P}$ by $A \leftarrow Body_1 \vee Body_2 \ldots$
- If $\mathcal{E}$ is not empty, replace all clauses of the form $A_1 \leftarrow Body_1,\ A_2 \leftarrow Body_2,\ \ldots$ occurring in $\mathcal{P}$ with $[A_1] = [A_2] = \ldots = [A]$ by $A \leftarrow Body_1 \vee Body_2 \ldots$

2 Add $A \leftarrow \bot$ for all undefined ground atoms A occurring in $\mathcal{P}$

3 Replace all occurrences of $\leftarrow$ by $\leftrightarrow$

- The resulting set is called completion of $\mathcal{P}$ or $c\,\mathcal{P}$
- If step 2 is omitted then the resulting set is called weak completion of $\mathcal{P}$ or $wc\,\mathcal{P}$

Program Completion – Example

- Let $\mathcal{P}$ be

$$
\begin{array}{rcl}
\ell & \leftarrow & e \wedge \neg ab_e \\
\ell & \leftarrow & t \wedge \neg ab_t \\
e & \leftarrow & \top \\
ab_e & \leftarrow & \bot \\
ab_t & \leftarrow & \bot
\end{array}
$$

- The weak completion of $\mathcal{P}$ consists of

$$
\begin{array}{rcl}
\ell & \leftrightarrow & (e \wedge \neg ab_e) \vee (t \wedge \neg ab_t) \\
e & \leftrightarrow & \top \\
ab_e & \leftrightarrow & \bot \\
ab_t & \leftrightarrow & \bot
\end{array}
$$

- The completion of $\mathcal{P}$ is obtained by adding

$$
t \leftrightarrow \bot
$$

Program Completion – Another Example

- Let $\mathcal{P}$ be

$$\begin{array}{lcl} p\,a & \leftarrow & \top \\ q\,b & \leftarrow & r\,b \end{array}$$

- How does $c\,\mathcal{P}$ look like?
- How does $wc\,\mathcal{P}$ look like?

Program Completion – Yet Another Example

- Let $\mathcal{P}$ be

$$\begin{array}{rcl} \ell & \leftarrow & e \wedge \neg ab_e \\ \ell & \leftarrow & o \wedge \neg ab_o \\ e & \leftarrow & \top \\ ab_e & \leftarrow & \bot \\ ab_o & \leftarrow & \bot \\ ab_e & \leftarrow & \neg o \\ ab_o & \leftarrow & \neg e \end{array}$$

- The weak completion of $\mathcal{P}$ consists of

$$\begin{array}{rcl} \ell & \leftrightarrow & (e \wedge \neg ab_e) \vee (o \wedge \neg ab_o) \\ e & \leftrightarrow & \top \\ ab_e & \leftrightarrow & \bot \vee \neg o \\ ab_o & \leftrightarrow & \bot \vee \neg e \end{array}$$

- Under Łukasiewicz logic we find $F \vee \bot \equiv F$

Convention

- Let $\mathcal{P}$ be a ground program and $\mathcal{E}$ an equational theory
- In the future
  - If $\mathcal{E}$ is empty, we will delete an assumption $A \leftarrow \bot$ if the program contains a fact $A \leftarrow \top$ or a rule $A \leftarrow Body$
  - If $\mathcal{E}$ is not empty, then $A \leftarrow \bot$ will be deleted if the ground program contains $B \leftarrow \top$ or $B \leftarrow Body$ with $[A] = [B]$

Sets of Literals versus Sets of Facts and Assumptions

- Let $\mathcal{S}$ be a consistent set of ground literals
- $\mathcal{S}^{\uparrow} = \{A \leftarrow \top \mid A \in \mathcal{S}\} \cup \{A \leftarrow \bot \mid \neg A \in \mathcal{S}\}$
- Let $\mathcal{P}$ be a ground program containing only facts and assumptions
  - Remember our convention!
- $\mathcal{P}^{\downarrow} = \{A \mid A \leftarrow \top \in \mathcal{P}\} \cup \{\neg A \mid A \leftarrow \bot \in \mathcal{P}\}$
- Example Let $\mathcal{S} = \{e, \neg ab_e\}$ and $\mathcal{P} = \{e \leftarrow \top,\ ab_e \leftarrow \bot\}$
  - $\mathcal{S}^{\uparrow} = \mathcal{P}$
  - $\mathcal{P}^{\downarrow} = \mathcal{S}$
- Is $\mathcal{P}^{\downarrow}$ consistent?

The Depends On Relation

- Let $\mathcal{P}$ be a ground program
- Atom A directly depends on atom B if
  - $\mathcal{P}$ contains a rule of the form $A \leftarrow Body$ and
  - B occurs (positively or negatively) in $Body$
- The depends on relation is the transitive closure of the directly depends on relation
- Example Let $\mathcal{P} = \{q\,a \leftarrow \top,\; q\,s\,a \leftarrow q\,a,\; q\,s\,s\,a \leftarrow q\,s\,a,\; \ldots\}$
  - $q\,s\,a$ directly depends on $q\,a$
  - $q\,s\,s\,a$ directly depends on $q\,s\,a$
  - $q\,s\,a$ depends on $q\,a$
  - $q\,s\,s\,a$ depends on $q\,s\,a$ and $q\,a$

The Function *deps*

- Let $\mathcal{P}$ be a ground program and $\mathcal{S}$ a consistent set of ground literals

$$deps(\mathcal{P}, \mathcal{S}) = \{B \leftarrow Body \in \mathcal{P} \mid Body \in \{\top, \bot\} \text{ and there exists } A \in \mathcal{S} \text{ or } \neg A \in \mathcal{S} \text{ such that } A \text{ depends on } B\}$$

- **Example** Let $\mathcal{P} = \{q\,a \leftarrow \top,\ q\,s\,a \leftarrow q\,a,\ q\,s\,s\,a \leftarrow q\,s\,a,\ \ldots\}$
  - $deps(\mathcal{P}, \{q\,s\,a\,a,\ \neg\, q\,s\,a\}) = \{q\,a \leftarrow \top\}$

The Meaning of Programs

- Let $\mathcal{P}$ be a program and $\mathcal{E}$ an equational theory
  - In many scenarios $\mathcal{E} = \emptyset$
  - When modeling ethical decision problems $\mathcal{E} = AC1$
- Recall equations, equational theories, interpretations, and models
- What is the meaning of $\mathcal{P}$?

Łukasiewicz Three-Valued Logic

F	$\neg F$
$\top$	$\bot$
$\bot$	$\top$
U	U

$\wedge$	$\top$	U	$\bot$
$\top$	$\top$	U	$\bot$
U	U	U	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$

$\vee$	$\top$	U	$\bot$
$\top$	$\top$	$\top$	$\top$
U	$\top$	U	U
$\bot$	$\top$	U	$\bot$

$\leftarrow$	$\top$	U	$\bot$
$\top$	$\top$	$\top$	$\top$
U	U	$\top$	$\top$
$\bot$	$\bot$	U	$\top$

$\leftrightarrow$	$\top$	U	$\bot$
$\top$	$\top$	U	$\bot$
U	U	$\top$	U
$\bot$	$\bot$	U	$\top$

Kleene Three-Valued Logic

F	$\neg F$
$\top$	$\bot$
$\bot$	$\top$
U	U

$\wedge$	$\top$	U	$\bot$
$\top$	$\top$	U	$\bot$
U	U	U	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$

$\vee$	$\top$	U	$\bot$
$\top$	$\top$	$\top$	$\top$
U	$\top$	U	U
$\bot$	$\top$	U	$\bot$

$\leftarrow$	$\top$	U	$\bot$
$\top$	$\top$	$\top$	$\top$
U	U	**U**	$\top$
$\bot$	$\bot$	U	$\top$

$\leftrightarrow$	$\top$	U	$\bot$
$\top$	$\top$	U	$\bot$
U	U	**U**	U
$\bot$	$\bot$	U	$\top$

Fitting Three-Valued Logic

F	$\neg F$
$\top$	$\bot$
$\bot$	$\top$
U	U

$\wedge$	$\top$	U	$\bot$
$\top$	$\top$	U	$\bot$
U	U	U	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$

$\vee$	$\top$	U	$\bot$
$\top$	$\top$	$\top$	$\top$
U	$\top$	U	U
$\bot$	$\top$	U	$\bot$

$\leftarrow$	$\top$	U	$\bot$
$\top$	$\top$	$\top$	$\top$
U	U	U	$\top$
$\bot$	$\bot$	U	$\top$

$\leftrightarrow$	$\top$	U	$\bot$
$\top$	$\top$	$\bot$	$\bot$
U	$\bot$	$\top$	$\bot$
$\bot$	$\bot$	$\bot$	$\top$

Three-Valued Interpretations

- A **(three-valued) interpretation** assigns to each formula a value from $\{\top, \bot, \mathsf{U}\}$
- It is represented by $\langle I^\top, I^\bot \rangle$, where
  - $I^\top$ contains all ground atoms which are mapped to $\top$
  - $I^\bot$ contains all ground atoms which are mapped to $\bot$
  - $I^\top \cap I^\bot = \emptyset$
  - All ground atoms which occur neither in $I^\top$ nor $I^\bot$ are mapped to U
- In the sequel, I, J denote interpretations $\langle I^\top, I^\bot \rangle$, $\langle J^\top, J^\bot \rangle$, respectively
- The **intersection** $I \cap J$ is defined as $\langle I^\top \cap J^\top, I^\bot \cap J^\bot \rangle$

Three-Valued Interpretations – Examples

- **Consider**

$\mathcal{P}$			$wc\,\mathcal{P}$			$c\,\mathcal{P}$		
ℓ	$\leftarrow$	$e \wedge \neg ab_e$	ℓ	$\leftrightarrow$	$(e \wedge \neg ab_e)$	ℓ	$\leftrightarrow$	$(e \wedge \neg ab_e)$
ℓ	$\leftarrow$	$t \wedge \neg ab_t$			$\vee\ (t \wedge \neg ab_t)$			$\vee\ (t \wedge \neg ab_t)$
e	$\leftarrow$	$\top$	e	$\leftrightarrow$	$\top$	e	$\leftrightarrow$	$\top$
ab_e	$\leftarrow$	$\bot$	ab_e	$\leftrightarrow$	$\bot$	ab_e	$\leftrightarrow$	$\bot$
ab_t	$\leftarrow$	$\bot$	ab_t	$\leftrightarrow$	$\bot$	ab_t	$\leftrightarrow$	$\bot$
						t	$\leftrightarrow$	$\bot$

- **Then**

I	$I\,\mathcal{P}$	$I\,wc\,\mathcal{P}$	$I\,c\,\mathcal{P}$
$\langle\{e, ab_e\}, \emptyset\rangle$	$\top$	$\bot$	$\bot$
$\langle\{e, \ell\}, \{ab_e, ab_t\}\rangle$	$\top$	$\top$	U
$\langle\{e, \ell, t\}, \{ab_e, ab_t\}\rangle$	$\top$	$\top$	$\bot$
$\langle\{e, \ell\}, \{ab_e, ab_t, t\}\rangle$	$\top$	$\top$	$\top$

Models

- An interpretation I is a **model** for a program $\mathcal{P}$ ($I \models \mathcal{P}$) iff $I\,\mathcal{P} = \top$
- This definition depends on the underlying logic!
  - We will indicate the underlying logic by adding a subscript to $\models$
  - Ł denotes Łukasiewicz logic
  - K denotes Kleene logic
  - F denotes Fitting logic
- Which of the following interpretations are models for

$$\mathcal{P} = \{a \leftarrow b\}$$

  - $\langle \emptyset, \emptyset \rangle \overset{?}{\models}_{Ł} \mathcal{P}$ $\quad \langle \{a, b\}, \emptyset \rangle \overset{?}{\models}_{Ł} \mathcal{P}$ $\quad \langle \emptyset, \{a, b\} \rangle \overset{?}{\models}_{Ł} \mathcal{P}$
  - $\langle \emptyset, \emptyset \rangle \overset{?}{\models}_{K} \mathcal{P}$ $\quad \langle \{a, b\}, \emptyset \rangle \overset{?}{\models}_{K} \mathcal{P}$ $\quad \langle \emptyset, \{a, b\} \rangle \overset{?}{\models}_{K} \mathcal{P}$
- In the sequel, we use Łukasiewicz logic if not stated otherwise

Model Intersection Property

- We would like to show that

$$\cap\{I \mid I \models \mathcal{P}\} \models \mathcal{P}$$

- It holds in classical two-valued logic for definite programs
- But it does not hold in classical two-valued logic for normal programs
- Under Łukasiewicz logic
  - The intersection of two models is not necessarily a model
  - Let $\mathcal{P}$ be the definite program

$$\begin{array}{lcl} p & \leftarrow & q_1 \wedge r_1 \\ p & \leftarrow & q_2 \wedge r_2 \end{array}$$

  - $\langle \emptyset, \{p, q_1, r_2\} \rangle \models \mathcal{P}$
  - $\langle \emptyset, \{p, q_2, r_1\} \rangle \models \mathcal{P}$
  - But $\langle \emptyset, \{p\} \rangle \not\models \mathcal{P}$

The Meaning of Programs

- **Proposition 10** If $I = \langle I^\top, I^\bot \rangle \models \mathcal{P}$ then $I' = \langle I^\top, \emptyset \rangle \models \mathcal{P}$
- **Proof** Suppose $I \models \mathcal{P}$,
  i.e., for all $A \leftarrow Body \in g\mathcal{P}$ we find $I \models A \leftarrow Body$
  - We consider the truth ordering $\bot <_t \mathsf{U} <_t \top$
  - We consider all cases for $I\, A$
  - We will show $I' \models A \leftarrow Body$ by $I'\, A \geq_t I'\, Body$
  - We distinguish three cases
    1. $I\, A = \top$ In this case $A \in I^\top$ and hence $I' \models A \leftarrow Body$
    2. $I\, A = \bot$
    3. $I\, A = \mathsf{U}$

Proof of Proposition 10 Case 2

2 $I\,A = \bot$ In this case $A \in I^{\bot}$ and $I'\,A = U$

- ▷ Because $I \models A \leftarrow Body$ we conclude $I\,Body = \bot$
- ▷ Hence we find a literal $L \in Body$ such that $I\,L = \bot$
  - ▶▶ $L = B$ In this case $I\,B = \bot$ and hence $I'\,B = I'\,L = U$
  - ▶▶ $L = \neg B$ In this case $I\,B = \top$ and hence $I'\,B = \top$ and $I'\,L = \bot$
- ▷ Consequently $I'\,Body \in \{U, \bot\}$
- ▷ Because $I'\,A = U$ we conclude $I' \models A \leftarrow Body$

Proof of Proposition 10 Case 3

3 $I\,A = \mathsf{U}$ In this case $I'\,A = \mathsf{U}$

- ▷ $I\,Body = \bot$ As in the previous case we find $I'\,Body \in \{\bot, \mathsf{U}\}$
  - ▶▶ Consequently $I' \models A \leftarrow Body$
- ▷ $I\,Body = \mathsf{U}$ In this case we find a literal $L \in Body$ with $I\,L = \mathsf{U}$
  - ▶▶ Then $I'\,L = \mathsf{U}$
  - ▶▶ Consequently $I'\,Body = \mathsf{U}$
  - ▶▶ Hence $I' \models A \leftarrow Body$ □

Proposition 10 – Examples

- Let $\mathcal{P} = \{\ell \leftarrow e \wedge \neg ab_e,\ e \leftarrow \top,\ ab_e \leftarrow \bot\}$
  - $\langle\{e, \ell\}, \{ab_e\}\rangle \models \mathcal{P}$
  - $\langle\{e, \ell\}, \emptyset\rangle \models \mathcal{P}$
- Let $\mathcal{E} = \{a \approx b\}$ and $\mathcal{P} = \{q\,X \leftarrow \neg\,p\,X,\ p\,a \leftarrow \top\}$
  - $\langle\{[p\,a]\}, \{[q\,b]\}\rangle \models \mathcal{P}$
  - $\langle\{[p\,a]\}, \emptyset\rangle \models \mathcal{P}$
- **Does Proposition 10 hold under Kleene or Fitting logic?**

Intersection of Two Models with Empty $\bot$-Part

- **Proposition 11** Let $I_1 = \langle I_1^\top, \emptyset\rangle$ and $I_2 = \langle I_2^\top, \emptyset\rangle$ be two models of $\mathcal{P}$
  Then $I = \langle I_1^\top \cap I_2^\top, \emptyset\rangle$ is also a model of $\mathcal{P}$
- **Proof** Suppose $I \not\models \mathcal{P}$
  - Then we find $A \leftarrow Body \in g\mathcal{P}$ such that $I \not\models A \leftarrow Body$
  - We distinguish three cases
    1. $I\,A = \bot$ and $I\,Body = \top$ Impossible because $I^\bot = \emptyset$
    2. $I\,A = \bot$ and $I\,Body = U$ Impossible because $I^\bot = \emptyset$
    3. $I\,A = U$ and $I\,Body = \top$

       Because $I\,A = U$ we find $j \in \{1, 2\}$ with $I_j\,A = U$

       Because $I_j \models A \leftarrow Body$ we find $I_j\,Body \in \{U, \bot\}$ (*)

       Because $I\,Body = \top$ and $I^\bot = \emptyset$ we find
       for all $L \in Body$ that L is an atom and $L \in I^\top$

       Hence for all $L \in Body$ we find $L \in I_j^\top$, $j \in \{1, 2\}$

       Consequently $I_j\,Body = \top$, $j \in \{1, 2\}$ contradicting (*) □

Model Intersection

- **Theorem 12** The model intersection property holds for $\mathcal{P}$ i.e., $\cap\{I \mid I \models \mathcal{P}\} \models \mathcal{P}$
- **Proof** Follows immediately from Propositions 10 and 11 $\square$
- **Example** Consider $\mathcal{P} = \{p \leftarrow q\}$
  - The least model of $\mathcal{P}$ under Łukasiewicz logic is $\langle\emptyset, \emptyset\rangle$
- Theorem 12 does not hold under Fitting logic ($\models_F$)
  - $\langle\{p, q\}, \emptyset\rangle \models_F p \leftarrow q$
  - $\langle\emptyset, \{p, q\}\rangle \models_F p \leftarrow q$
  - However $\langle\emptyset, \emptyset\rangle \not\models_F p \leftarrow q$
- Theorem 12 does not hold under Kleene logic ($\models_K$)
- What are the least models for the first three programs in the suppression task?

The Meaning of Weakly Completed Programs

- **Theorem 13** The model intersection property holds for $wc\,\mathcal{P}$ as well
- **Proof** later in the lecture
- $\mathcal{M}_{wc\mathcal{P}}$ denotes the least model of $wc\,\mathcal{P}$
- **Is $\mathcal{M}_{wc\mathcal{P}}$ the least model of $\mathcal{P}$?**
- **Corollary 14** If $I \models wc\,\mathcal{P}$ then $I \models \mathcal{P}$
- **Proof** $F \leftrightarrow G \equiv (F \rightarrow G) \wedge (G \rightarrow F)$ under Łukasiewicz logic $\square$
- Proposition 14 does not hold under Fitting logic
  - $\langle \emptyset, \emptyset \rangle \models_F wc\{p \leftarrow q\} = \{p \leftrightarrow q\}$
  - However $\langle \emptyset, \emptyset \rangle \not\models_F \{p \leftarrow q\}$

The Suppression Task – Experiments 1-3

Ex.	$\mathcal{P}$	$wc\,\mathcal{P}$	$\mathcal{M}_{wc\mathcal{P}}$
1	$e \leftarrow \top$ $\ell \leftarrow e \wedge \neg ab_e$ $ab_e \leftarrow \bot$	$e \leftrightarrow \top$ $\ell \leftrightarrow e \wedge \neg ab_e$ $ab_e \leftrightarrow \bot$	$\langle\{e, \ell\}, \{ab_e\}\rangle$
2	$e \leftarrow \top$ $\ell \leftarrow e \wedge \neg ab_e$ $\ell \leftarrow t \wedge \neg ab_t$ $ab_e \leftarrow \bot$ $ab_t \leftarrow \bot$	$e \leftrightarrow \top$ $\ell \leftrightarrow (e \wedge \neg ab_e) \vee (t \wedge \neg ab_t)$ $ab_e \leftrightarrow \bot$ $ab_t \leftrightarrow \bot$	$\langle\{e, \ell\}, \{ab_e, ab_t\}\rangle$
3	$e \leftarrow \top$ $\ell \leftarrow e \wedge \neg ab_e$ $\ell \leftarrow o \wedge \neg ab_o$ $ab_e \leftarrow \bot$ $ab_o \leftarrow \bot$ $ab_e \leftarrow \neg o$ $ab_o \leftarrow \neg e$	$e \leftrightarrow \top$ $\ell \leftrightarrow (e \wedge \neg ab_e) \vee (o \wedge \neg ab_o)$ $ab_e \leftrightarrow \bot \vee \neg o$ $ab_o \leftrightarrow \bot \vee \neg e$	$\langle\{e\}, \{ab_o\}\rangle$

The Suppression Task – Experiments 4-6

Ex.	$\mathcal{P}$	$wc\,\mathcal{P}$	$\mathcal{M}_{wc\mathcal{P}}$
4	$e \leftarrow \bot$ $\ell \leftarrow e \wedge \neg ab_e$ $ab_e \leftarrow \bot$	$e \leftrightarrow \bot$ $\ell \leftrightarrow e \wedge \neg ab_e$ $ab_e \leftrightarrow \bot$	$\langle \emptyset, \{e, \ell, ab_e\} \rangle$
5	$e \leftarrow \bot$ $\ell \leftarrow e \wedge \neg ab_e$ $\ell \leftarrow t \wedge \neg ab_t$ $ab_e \leftarrow \bot$ $ab_t \leftarrow \bot$	$e \leftrightarrow \bot$ $\ell \leftrightarrow (e \wedge \neg ab_e) \vee (t \wedge \neg ab_t)$ $ab_e \leftrightarrow \bot$ $ab_t \leftrightarrow \bot$	$\langle \emptyset, \{e, ab_e, ab_t\} \rangle$
6	$e \leftarrow \bot$ $\ell \leftarrow e \wedge \neg ab_e$ $\ell \leftarrow o \wedge \neg ab_o$ $ab_e \leftarrow \bot$ $ab_o \leftarrow \bot$ $ab_e \leftarrow \neg o$ $ab_o \leftarrow \neg e$	$e \leftrightarrow \bot$ $\ell \leftrightarrow (e \wedge \neg ab_e) \vee (o \wedge \neg ab_o)$ $ab_e \leftrightarrow \bot \vee \neg o$ $ab_o \leftrightarrow \bot \vee \neg e$	$\langle \{ab_o\}, \{e, \ell\} \rangle$

Monotonicity

- Let $\mathcal{P}$ and $\mathcal{P}'$ be sets of formulas and G a formula
  A logic is monotonic if the following holds:
  If $\mathcal{P} \models G$ then $\mathcal{P} \cup \mathcal{P}' \models G$
- Classical logic is monotonic
- A logic based on the weak completion semantics is non-monotonic
  - Consider

$$\begin{array}{lcl} \mathcal{P} & = & \{c \leftarrow \bot\} \\ \mathcal{P}' & = & \{c \leftarrow \top\} \end{array}$$

  - Then

$$\begin{array}{lclcl} wc\,\mathcal{P} & = & \{c \leftrightarrow \bot\} & \models & \neg c \\ wc\,(\mathcal{P} \cup \mathcal{P}') & = & \{c \leftrightarrow \bot \vee \top\} & \models & c \end{array}$$

Computing Least Models

- **How can we compute the least models of weakly completed programs?**
- **In classical two-valued logic we obtain**

$$T_{\mathcal{P}}\ I = \{A \mid \text{there exists } A \leftarrow Body \in g\,\mathcal{P} \text{ with } I\ Body = \top\}$$

where $\mathcal{P}$ is a definite logic program and I an interpretation

- **In three-valued logic programming we obtain $\Psi_{\mathcal{P}}\ I = \langle J^{\top}, J^{\bot} \rangle$ where**

$$\begin{array}{lcl} J^{\top} & = & \{A \mid \text{there exists } A \leftarrow Body \in g\,\mathcal{P} \text{ with } I\ Body = \top\} \\ J^{\bot} & = & \{A \mid \text{for all } A \leftarrow Body \in g\,\mathcal{P} \text{ we find } I\ Body = \bot\} \end{array}$$

  - **$\Psi_{\mathcal{P}}$ is monotone on $(\mathcal{I}, \subseteq)$**
  - **The least model of $c\,\mathcal{P}$ under Fitting logic is the least fixed point of $\Psi_{\mathcal{P}}$**
  - **Inadequate for human reasoning $\rightsquigarrow$ Why?**

The Semantic Operator for Weakly Completed Programs

- **Consider the following immediate consequence operator**

  $\Phi'_{\mathcal{P}}\, I = \langle J^{\top}, J^{\bot} \rangle$ **where**

$$
\begin{array}{lcl}
J^{\top} & = & \{A \mid \text{there exists } A \leftarrow Body \in g\,\mathcal{P} \text{ with } I\,Body = \top\} \\
J^{\bot} & = & \{A \mid \text{there exists } A \leftarrow Body \in g\,\mathcal{P} \text{ and} \\
 & & \quad \text{for all } A \leftarrow Body \in g\,\mathcal{P} \text{ we find } I\,Body = \bot\}
\end{array}
$$

- $\Phi'_{\mathcal{P}}$ **"without the red condition" is** $\Psi_{\mathcal{P}}$

The Semantic Operator for Weakly Completed Programs with Equality

- Let $\mathcal{P}$ be a program, $\mathcal{E}$ an equational theory, and I an interpretation
- Consider the following immediate consequence operator
  $\Phi_{\mathcal{P}}\, I = \langle J^\top, J^\bot \rangle$ where

$$
\begin{array}{rcl}
J^\top & = & \{[A] \mid \text{there exists } A \leftarrow Body \in g\,\mathcal{P} \text{ with } I\,Body = \top\} \\
J^\bot & = & \{[A] \mid \text{there exists } A \leftarrow Body \in g\,\mathcal{P} \text{ and} \\
 & & \quad \text{for all } A' \leftarrow Body \in g\,\mathcal{P} \text{ with } [A] = [A'] \text{ we find } I\,Body = \bot\}
\end{array}
$$

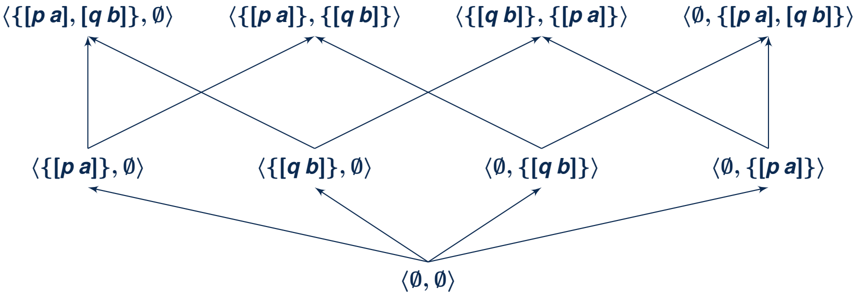
  and $[A]$ denotes the finest congruence class defined by $\mathcal{E}$ and containing A

Semantic Operator – Examples

- Iteratively apply $\Phi_{\mathcal{P}}$ to the following programs starting with $\langle \emptyset, \emptyset \rangle$
  - $\mathcal{P} = \{e \leftarrow \top,\ \ell \leftarrow e \wedge \neg ab_e,\ ab_e \leftarrow \bot\}$ and $\mathcal{E} = \emptyset$
  - $\mathcal{P} = \{q\,X \leftarrow \neg p\,X,\ p\,a \leftarrow \top\}$ and $\mathcal{E} = \{a \approx b\}$
- Do least fixed points of $\Phi_{\mathcal{P}}$ always exist?
- How long does it take to compute least fixed points of $\Phi_{\mathcal{P}}$?
  - Recall fixed point theory

The Complete Partial Order of Interpretations – Example

- Let $\mathcal{P} = \{p\,X \leftarrow q\,X\}$ and $\mathcal{E} = \{a \approx b\}$
- Let $\mathcal{I}$ denote the set of all three-valued interpretations
- $I = \langle I^\top, I^\bot \rangle \subseteq \langle J^\top, J^\bot \rangle = J$ iff $I^\top \subseteq J^\top$ and $I^\bot \subseteq J^\bot$
- $(\mathcal{I}, \subseteq)$ is a complete partial order



The Complete Partial Order of Interpretations 1

- Let $\mathcal{P}$ be a program and $\mathcal{E}$ an equational theory
- Let $\mathcal{J}$ be a set of interpretations
  - $\mathcal{J}^\top = \{I^\top \mid \langle I^\top, I^\bot\rangle \in \mathcal{J}\}$
  - $\mathcal{J}^\bot = \{I^\bot \mid \langle I^\top, I^\bot\rangle \in \mathcal{J}\}$
- **Proposition 15** Let $\mathcal{J}$ be a directed set of interpretations
  Then the interpretation $I = \langle \bigcup \mathcal{J}^\top, \bigcup \mathcal{J}^\bot\rangle$ is the least upper bound of $\mathcal{J}$
- **Proof** Given $\mathcal{J}$ we have to show that

  (i) I is an interpretation

  (ii) I is an upper bound of $\mathcal{J}$ $\rightsquigarrow$ Exercise

  (iii) I is the least upper bound of $\mathcal{J}$ $\rightsquigarrow$ Exercise

Proof of Proposition 15 (i)

- To show $I = \langle \bigcup \mathcal{J}^\top, \bigcup \mathcal{J}^\bot \rangle$ is an interpretation
  - By definition $\bigcup \mathcal{J}^\top$ and $\bigcup \mathcal{J}^\bot$ are unions of congruence classes
  - It remains to show $\bigcup \mathcal{J}^\top \cap \bigcup \mathcal{J}^\bot = \emptyset$
  - Suppose we find $[A] \in \bigcup \mathcal{J}^\top \cap \bigcup \mathcal{J}^\bot$
  - Then we find $I_1, I_2 \in \mathcal{J}$ with $[A] \in I_1^\top$ and $[A] \in I_2^\bot$
  - Because $\mathcal{J}$ is directed, it contains a common upper bound K of I_1 and I_2
  - We find $[A] \in K^\top$ and $[A] \in K^\bot$
  - Hence, K cannot be an interpretation $\leadsto$ contradiction $\square$

The Complete Partial Order of Interpretations 2

- **Corollary 16**
  The set of all interpretations $\mathcal{I}$ is a complete partial order with respect to $\subseteq$
- **Proof**
  - Reflexivity, antisymmetry, and transitivity holds for $\subseteq$
  - By Proposition 15 every directed subset of $\mathcal{I}$ has a least upper bound in $\mathcal{I}$

Monotonocity of the Semantic Operator

- **Proposition 17**
  For each program $\mathcal{P}$ and equational theory $\mathcal{E}$ the mapping $\Phi_{\mathcal{P}}$ is monotonic
- **Proof** Let $I = \langle I^\top, I^\bot \rangle \subseteq \langle J^\top, J^\bot \rangle = J$
  - To show $\Phi_{\mathcal{P}}\, I = I' = \langle I'^\top, I'^\bot \rangle \subseteq \langle J'^\top, J'^\bot \rangle = J' = \Phi_{\mathcal{P}}\, J$
  - $I'^\top \subseteq J'^\top$
    - $[A] \in I'^\top$ iff we find $A \leftarrow Body \in g\,\mathcal{P}$ such that $I\, Body = \top$
    - Because $I \subseteq J$ we claim $J\, Body = \top$ prove it!
    - Hence, $[A] \in J'^\top$
  - $I'^\bot \subseteq J'^\bot$ $\rightsquigarrow$ Exercise $\square$

Non-Continuity of the Semantic Operator 1

- Let $\mathcal{E} = \emptyset$ and $\mathcal{P}$ be

$$\begin{array}{rcl} q\,a & \leftarrow & \top \\ q\,s\,X & \leftarrow & q\,X \\ p & \leftarrow & \neg\, q\,X \end{array}$$

- The least fixed point of $\Phi_{\mathcal{P}}$ is

$$\langle \{[q\, s^k\, a] \mid k \in \mathbb{N}\}, \{[p]\}\rangle$$

- It is reached after $\omega + 1$ iterations
- By the Kleene Fixed Point Theorem 4 $\Phi_{\mathcal{P}}$ is not continuous
- The Herbrand base contains infinitely many equivalence classes

$$[p],\ [q\,a],\ [q\,s\,a],\ \ldots$$

where each equivalence class has one member

Non-Continuity of the Semantic Operator 2

- Let $\mathcal{P}$ be

$$\begin{array}{rcl} q\ 1 & \leftarrow & \top \\ q\ (X \circ a) & \leftarrow & q\ X \\ p & \leftarrow & \neg\ q\ X \end{array}$$

and $\mathcal{E}$ be

$$\begin{array}{rcl} X \circ (Y \circ Z) & \approx & (X \circ Y) \circ Z \\ X \circ Y & \approx & Y \circ X \\ X \circ 1 & \approx & X \end{array}$$

- The least fixed point of $\Phi_{\mathcal{P}}$ is

$$\langle \{[q(1 \circ \overbrace{a \circ \ldots \circ a}^{k})] \mid k \in \mathbb{N}\}, \{[p]\} \rangle$$

- It is reached after $\omega + 1$ iterations
- By Kleene Fixed Point Theorem 4 $\Phi_{\mathcal{P}}$ is not continuous
- The Herbrand base contains infinitely many equivalence classes

$$[p],\ [q\ 1],\ [q\ a],\ [q(a \circ a)],\ \ldots$$

where with the exception of $[p]$ each of these equivalence classes is infinite

Finite Propositional and Finite Ground Programs

- **Proposition 18**
  For each finite propositional program $\mathcal{P}$ the mapping $\Phi_{\mathcal{P}}$ is continuous
- **Proof**
  - Because $\mathcal{P}$ is finite, the set $\mathcal{I}$ of interpretations is finite
  - By Corollary 16 $(\mathcal{I}, \subseteq)$ is a complete partial order
  - By Proposition 17 $\Phi_{\mathcal{P}}$ is monotonic on $\mathcal{I}$
  - By Proposition 7 the mapping $\Phi_{\mathcal{P}}$ is continuous □
- **Proposition 19**
  If the Herbrand base for a program $\mathcal{P}$ and a set of equations $\mathcal{E}$ is finite then the mapping $\Phi_{\mathcal{P}}$ is continuous
- **Proof**
  - Define a bijection between the set of ground atoms occurring in $\mathcal{P}$ and an equally large set of propositional atoms
  - Replace each ground atom by a propositional atom
  - Apply Proposition 18 □

Least Fixed Points and Models

- **Lemma 20** Let J be the least fixed point of $\Phi_{\mathcal{P}}$ and I a model of $wc\mathcal{P}$
  - Then for every ground atom A we find
    - If $J\,A = \top$ then $I\,A = \top$
    - If $J\,A = \bot$ then $I\,A = \bot$
- **Proof** Let J be the least fixed point of $\Phi_{\mathcal{P}}$ and I a model of $wc\mathcal{P}$
  - We start iterating $\Phi_{\mathcal{P}}$ on $\langle\emptyset, \emptyset\rangle$
  - **Claim** For every ordinal α and every ground atom A we find
    - If $\Phi_{\mathcal{P}} \uparrow \alpha\, A = \top$ then $I\,A = \top$
    - If $\Phi_{\mathcal{P}} \uparrow \alpha\, A = \bot$ then $I\,A = \bot$
  - **Proof of the Claim** by transfinite induction $\rightsquigarrow$ **Exercise**
  - The lemma follows from Propositions 3 and 17 $\square$

Lemma 20 – Example

- **Let** $\mathcal{P} = \{q\,a \leftarrow \top,\ q\,s\,X \leftarrow q\,X,\ p \leftarrow \neg\,q\,X,\ r\,a \leftarrow \top\}$
- $I = \langle\{q\,s^k\,a \mid k \in \mathbb{N}\} \cup \{r\,a,\ r\,s^2\,a\}, \{p,\ r\,s\,a\}\rangle$ **is a model of** $wc\,\mathcal{P}$

$$
\begin{array}{rl}
\Phi_{\mathcal{P}} \uparrow 0 & \langle\emptyset, \emptyset\rangle \\
\Phi_{\mathcal{P}} \uparrow 1 & \langle\{q\,a,\ r\,a\}, \emptyset\rangle \\
\Phi_{\mathcal{P}} \uparrow 2 & \langle\{q\,a,\ q\,s\,a,\ r\,a\}, \emptyset\rangle \\
\vdots & \vdots \\
\Phi_{\mathcal{P}} \uparrow \omega & \langle\{q\,s^k\,a \mid k \in \mathbb{N}\} \cup \{r\,a\}, \emptyset\rangle \\
\Phi_{\mathcal{P}} \uparrow (\omega + 1) & \langle\{q\,s^k\,a \mid k \in \mathbb{N}\} \cup \{r\,a\}, \{p\}\rangle
\end{array}
$$

Fixed Points are Models

- **Lemma 21**
  If I is a fixed point of $\Phi_{\mathcal{P}}$ then I is a model of $wc\,\mathcal{P}$
- **Proof** to show $I(A \leftrightarrow F) = \top$ for all $A \leftrightarrow F \in wc\,\mathcal{P}$
  - $[A] \in I^{\top}$ We find $A \leftarrow Body \in \mathcal{P}$ with $I\,Body = \top$
    - Then, $F = Body \vee F'$ and $I\,F = \top$
    - Hence, $I\,A = I\,F$
  - $[A] \in I^{\bot}$ $\leadsto$ Exercise
  - $[A] \not\in I^{\top} \cup I^{\bot}$ $\leadsto$ Exercise $\square$

Least Fixed Points are Minimal Models

- **Proposition 22**
  If J is the least fixed point of $\Phi_{\mathcal{P}}$ then J is a minimal model of $wc\,\mathcal{P}$
- **Proof** Let J be the least fixed point of $\Phi_{\mathcal{P}}$
  - By Lemma 21 J is a model of $wc\,\mathcal{P}$
  - By Proposition 20 for every model I of $wc\mathcal{P}$ we find $J^{\top} \subseteq I^{\top}$ and $J^{\bot} \subseteq I^{\bot}$, i.e., $J \subseteq I$
  - Hence, J is a minimal model of $wc\,\mathcal{P}$ $\square$

Least Fixed Points and Least Models

- **Proposition 23**
  If I is a minimal model of $wc\,\mathcal{P}$ then I is the least fixed point of $\Phi_{\mathcal{P}}$
- **Proof** Let I be a minimal model of $wc\mathcal{P}$ and J be the least fixed point of $\Phi_{\mathcal{P}}$
  - From Lemma 20 we learn that $J^{\top} \subseteq I^{\top}$ and $J^{\bot} \subseteq I^{\bot}$
  - But then $I = J$ as otherwise we have a conflict with the minimality of I $\square$
- **Theorem 13** $wc\,\mathcal{P}$ has a least model
- **Proof** Follows from Propositions 22 and 23 and the fact that the least fixed point of $\Phi_{\mathcal{P}}$ is unique $\square$
- **Theorem 24** I is the least fixed point of $\Phi_{\mathcal{P}}$ **iff** I is the least model of $wc\,\mathcal{P}$
- **Proof** Follows from Theorem 13 and Propositions 22 and 23 $\square$

Entailment under the Weak Completion Semantics

- Let $\mathcal{M}_{wc\mathcal{P}}$ denote the least fixed point of $\Phi_{\mathcal{P}}$
  - which is equal to the least model of $wc\,\mathcal{P}$
- $\mathcal{P}$ entails F under the weak completion semantics

$$\mathcal{P} \models_{wcs} F \quad \text{iff} \quad \mathcal{M}_{wc\mathcal{P}}\, F = \top$$

Two Examples

- Consider the program $\mathcal{P} = \{p \leftarrow q,\ q \leftarrow p\}$
  - It has a least model $\langle\emptyset, \emptyset\rangle$
  - It can be computed iterating $\Phi_{\mathcal{P}}$ starting with $\langle\emptyset, \emptyset\rangle$
  - But if the iteration would start with $\langle\{p\}, \emptyset\rangle$ then it will run forever
  - Do humans always start with the empty interpretation?
- Consider the program $\mathcal{P} = \{even\ 0 \leftarrow \top,\ even\ s\,X \leftarrow \neg\, even\ X\}$
  - It has a least model $\langle\{even\ s^k\ 0 \mid k \text{ is even}\}, \{even\ s^k\ 0 \mid k \text{ is odd}\}\rangle$
  - It can be computed iterating $\Phi_{\mathcal{P}}$ starting with $\langle\emptyset, \emptyset\rangle$
  - How many steps do we need?
- We will address both questions using metric methods

Semantic Operators as Contraction Mappings

- A level mapping for $\mathcal{P}$ is a mapping *level* from the set of ground atoms to $\mathbb{N}$ such that $level\ A = level\ B$ iff $[A] = [B]$
  - It is extended to a mapping from ground literals to $\mathbb{N}$ by $level\ \neg A = level\ A$
- Let *level* be a level mapping for $\mathcal{P}$
  - $\mathcal{P}$ is acyclic with respect to *level* iff
    for every rule $A \leftarrow L_1 \wedge \ldots \wedge L_n \in g\,\mathcal{P}$
    we find $level\ A > level\ L_i$ for all $1 \leq i \leq n$
  - $\mathcal{P}$ is acyclic iff it is acyclic with respect to some level mapping
  - The problem to determine whether $\mathcal{P}$ is acyclic is undecidable

Acyclic Programs – Examples 1

- Consider the program $\mathcal{P}$

$$\begin{array}{lcl} p & \leftarrow & r \wedge q \\ q & \leftarrow & r \wedge p \end{array}$$

  - Is $\mathcal{P}$ acyclic?
  - How many fixed points has $\Phi_{\mathcal{P}}$?
  - Is $\Phi_{\mathcal{P}}$ a contraction on a complete metric space?
- Are the followig programs acyclic?
  - $\{q\ a \leftarrow \top,\ q\ s\ X \leftarrow q\ X,\ p \leftarrow \neg\, q\ X\}$
  - $\{even\ 0 \leftarrow \top,\ even\ s\ X \leftarrow \neg\, even\ X\}$

Acyclic Programs – Examples 2

- Consider the program $\mathcal{P}$

$$\begin{array}{lcl} p & \leftarrow & q \wedge r \\ q & \leftarrow & \neg r \\ r & \leftarrow & \top \end{array}$$

- Let *level* $r = 0$, *level* $q = 1$, *level* $p = 2$
  - $\mathcal{P}$ is acyclic with respect to *level*
  - We find

$$\begin{array}{lcl} \Phi_{\mathcal{P}}(\langle\{q, r\}, \{p\}\rangle) & = & \langle\{p, r\}, \{q\}\rangle \\ \Phi_{\mathcal{P}}(\langle\{p, r\}, \{q\}\rangle) & = & \langle\{r\}, \{p, q\}\rangle \end{array}$$

$$\begin{array}{lcl} \Phi_{\mathcal{P}}(\langle\{p\}, \emptyset\rangle) & = & \langle\{r\}, \emptyset\rangle \\ \Phi_{\mathcal{P}}(\langle\{r\}, \emptyset\rangle) & = & \langle\{r\}, \{q\}\rangle \\ \Phi_{\mathcal{P}}(\langle\{r\}, \{q\}\rangle) & = & \langle\{r\}, \{p, q\}\rangle \end{array}$$

  - $\langle\{r\}, \{p, q\}\rangle$ is the unique fixed point of $\Phi_{\mathcal{P}}$
  - Is $\Phi_{\mathcal{P}}$ a contraction? If so, on what metric space?

Programs and Metric Spaces

- **Proposition 25** Let $\mathcal{P}$ be a program, $\mathcal{E}$ an equational theory, *level* a level mapping for $\mathcal{P}$, $\mathcal{I}$ the set of interpretations for $\mathcal{P}$, and $I, J \in \mathcal{I}$
  - The function $d_{level} : \mathcal{I} \times \mathcal{I} \to \mathbb{R}$ defined as

$$d_{level}(I, J) = \begin{cases} \frac{1}{2^n} & I \neq J \text{ and} \\ & I\,A = J\,A \neq \mathsf{U} \text{ for all } A \text{ with } level\ A < n \text{ and} \\ & I\,A \neq J\,A \text{ or } I\,A = J\,A = \mathsf{U} \text{ for some } A \text{ with } level\ A = n \\ 0 & \text{otherwise} \end{cases}$$

    is a metric

- **Proof** $\rightsquigarrow$ **Exercise**

Programs and Metric Spaces – Example 1

- Consider the program $\mathcal{P}$

$$\begin{array}{rcl} \textit{even } 0 & \leftarrow & \top \\ \textit{even } s\,X & \leftarrow & \neg\, \textit{even } X \end{array}$$

- Let

$$\begin{array}{rcl} I & = & \langle\{\textit{even } s^k\ 0 \mid k \in \{0, 2, \ldots\}\}, \{\textit{even } s^k\ 0 \mid k \in \{1, 3, \ldots\}\}\rangle \\ J & = & \langle\{\textit{even } s^k\ 0 \mid k \in \{0, 2, \ldots\}\}, \emptyset\rangle \end{array}$$

  and

$$\textit{level even } s^k\ 0 = k$$

- Then

$$d_{level}(I, J) = \frac{1}{2}$$

- **Note** $g\,\mathcal{P}$ is infinite and $\mathcal{P}$ is acyclic

Programs and Metric Spaces – Example 2

- Consider again the program $\mathcal{P}$

$$\begin{array}{rcl} \mathit{even}\ 0 & \leftarrow & \top \\ \mathit{even}\ s\ X & \leftarrow & \neg\, \mathit{even}\ X \end{array}$$

- Let again $\mathit{level}\ \mathit{even}\ s^k\ 0 = k$
- For all $n \in \mathbb{N}$ let

$$I_n = \langle \{\mathit{even}\ s^k\ 0 \mid k \leq n \text{ and } k \text{ even}\}, \{\mathit{even}\ s^k\ 0 \mid k \leq n \text{ and } k \text{ odd}\} \rangle$$

- What is the distance between I_n and I_m?
- Is the sequence $(I_n \mid n \geq 0)$ a Cauchy sequence?
- Does the sequence $(I_n \mid n \geq 0)$ converge?

Programs and Complete Metric Spaces

- Let *level* be a level mapping for $\mathcal{P}$, $\mathcal{E}$ an equational theory and $\mathcal{I}$ the set of interpretations for $\mathcal{P}$
- **Proposition 26** $(\mathcal{I}, d_{level})$ is a complete metric space
- **Proof** **To show** Every Cauchy sequence of interpretations converges
  - Let $(I_k \mid k \geq 1)$ be a Cauchy sequence of interpretations
  - I.e., for all $\varepsilon > 0$ there is $K \in \mathbb{N}$: for all $k_1, k_2 \geq K$ we find $d_{level}(I_{k_1}, I_{k_2}) \leq \varepsilon$
  - In particular, for all $n \in \mathbb{N}$, there is $K \in \mathbb{N}$: for all $k_1, k_2 \geq K$ we find

$$d_{level}(I_{k_1}, I_{k_2}) \leq \frac{1}{2^{n+1}}$$

  - For all $n \in \mathbb{N}$ let K_n be the least such K
  - Hence, if $n_1 \leq n_2$ then $\frac{1}{2^{n_1+1}} \geq \frac{1}{2^{n_2+1}}$ and $K_{n_1} \leq K_{n_2}$
  - **To show** $(I_k \mid k \geq 1)$ converges
  - i.e., there is I: for every $\varepsilon > 0$, there is a K: for all $k \geq K$ we find $d(I, I_k) \leq \varepsilon$

Proof of Proposition 26 – Continued

- Let I be such that for each ground atom A we have $I\,A = I_{K_\ell}\,A$ where $\ell = level\,A$
- We choose $\varepsilon > 0$ and let $n \in \mathbb{N}$ be such that $\frac{1}{2^{n+1}} \leq \varepsilon$
- **Claim** $d_{level}(I, I_k) \leq \frac{1}{2^{n+1}} \leq \varepsilon$ for any $k \geq K_n$
- **Proof of the Claim** $\rightsquigarrow$ **Exercise** $\square$

Programs and Contractions

- **Let *level* be a level mapping for $\mathcal{P}$, $\mathcal{E}$ an equational theory and $\mathcal{I}$ the set of interpretations for $\mathcal{P}$**
- **Theorem 27**
  If $\mathcal{P}$ is acyclic with respect to *level* then $\Phi_{\mathcal{P}}$ is a contraction on $(\mathcal{I}, d_{level})$
- **Proof** **we will prove a more general result later in the lecture**
- **Corollary 28** **If $\mathcal{P}$ is acyclic then $\Phi_{\mathcal{P}}$ has a unique fixed point which can be reached by iterating $\Phi_{\mathcal{P}}$ up to ω times starting with any interpretation**
- **Proof** **Follows from Theorems 27 and 9** □

Reconsidering Two Examples

- Reconsider the program $\mathcal{P} = \{p \leftarrow q,\ q \leftarrow p\}$
  - It is not acyclic
  - Model construction must start with the empty interpretation
- Reconsider the program $\mathcal{P} = \{even\ 0 \leftarrow \top,\ even\ s\ X \leftarrow \neg\ even\ X\}$
  - It is acyclic
  - Model construction can start with any interpretation

$\Phi_{\mathcal{P}}$	$I^{\top}$	$I^{\bot}$
$\uparrow 0$		*even 0*
$\uparrow 1$	*even 0* *even s 0*	
$\uparrow 2$	*even 0*	*even s 0* *even s s 0*
$\vdots$	$\vdots$	$\vdots$

  - The least fixed point will be computed in ω steps

Abduction – Overview

- Integrity constraints
- Abducibles
- Abductive Frameworks
- Observations
- Credulous versus skeptical reasoning
- Examples

Abduction

- Charles Sanders Peirce 1932
  - given a program and an observation (which is not entailed by the program)
  - a consistent set of facts (and assumptions) is infered or abduced
  - such that the program and the facts entail the observation
- The set of facts is called explanation for the observation
- Applications
  - fault diagnosis
  - high level vision
  - natural language processing
  - planning
  - knowledge assimilation
  - ...

Integrity Constraints

- **Integrity constraints** are formulas of the form

$$\mathsf{U} \leftarrow Body \text{ (weak IC)} \quad \text{or} \quad \bot \leftarrow Body \text{ (strong IC)}$$

  where *Body* is a conjunction of literals
- $\mathcal{IC}$ denotes a finite set of integrity constraints
- Interpretation I **satisfies** $\mathcal{IC}$ **iff** I satisfies each constraint occurring in $\mathcal{IC}$
- Integrity constraints eliminate models
- **Examples**

a	$\mathsf{U} \leftarrow a$	$\bot \leftarrow a$	$\mathsf{U} \leftarrow \neg a$	$\bot \leftarrow \neg a$
$\top$	U	$\bot$	$\top$	$\top$
U	$\top$	U	$\top$	U
$\bot$	$\top$	$\top$	U	$\bot$

- **What is the difference between $\bot \leftarrow a$ and $a \leftarrow \bot$?**

Integrity Constraints – Preferences

- **Michael believes that offering Kim a homemade cake or homemade cookies will make her happy. But he also knows that she does not want both.**

$$\begin{array}{rcl} happy & \leftarrow & cake \wedge \neg ab_{cake} \\ happy & \leftarrow & cookies \wedge \neg ab_{cookies} \\ ab_{cake} & \leftarrow & \bot \\ ab_{cookies} & \leftarrow & \bot \end{array}$$

cake	*cookies*	U ← *cake* ∧ *cookies*	⊥ ← *cake* ∧ *cookies*
⊤	⊤	U	⊥
⊤	U	⊤	U
⊤	⊥	⊤	⊤
U	⊤	⊤	U
U	U	⊤	U
U	⊥	⊤	⊤
⊥	⊤	⊤	⊤
⊥	U	⊤	⊤
⊥	⊥	⊤	⊤

Integrity Constraints and Models

- **Suppose $\mathcal{IC} \neq \emptyset$**
- **Then $\mathcal{P}$ as well as *wc* $\mathcal{P}$ may not have models satisfying $\mathcal{IC}$**
- **Can you specify an example?**

Abducibles

- Let $\mathcal{P}$ be a ground program
- The **set of abducibles** is

$$\mathcal{A}_\mathcal{P} = \{A \leftarrow \top \mid A \text{ is undefined in } \mathcal{P}\} \cup \{A \leftarrow \bot \mid A \text{ is undefined in } \mathcal{P}\}$$

- **Should defeaters of negative assumptions be added to this set?**

Abductive Frameworks

- Let $\mathcal{P}$ be a ground program
- An abductive framework $\langle \mathcal{P}, \mathcal{A}_{\mathcal{P}}, \mathcal{IC}, \models_{wcs} \rangle$ consists of
  - a program $\mathcal{P}$
  - a set of abducibles $\mathcal{A}_{\mathcal{P}}$
  - a set $\mathcal{IC}$ of integrity constraints
  - the entailment relation $\models_{wcs}$
- In the sequel, we sometimes consider datalog programs
  - In this case, the set of abducibles as well as abductive frameworks are defined with respect to the ground instances of the program

The Suppression Task – Abducibles

$\mathcal{P}$			$\mathcal{A}_{\mathcal{P}}$		
ℓ	$\leftarrow$	$e \wedge \neg ab_e$	e	$\leftarrow$	$\top$
ab_e	$\leftarrow$	$\bot$	e	$\leftarrow$	$\bot$
ℓ	$\leftarrow$	$e \wedge \neg ab_e$	e	$\leftarrow$	$\top$
ℓ	$\leftarrow$	$t \wedge \neg ab_t$	e	$\leftarrow$	$\bot$
ab_e	$\leftarrow$	$\bot$	t	$\leftarrow$	$\top$
ab_t	$\leftarrow$	$\bot$	t	$\leftarrow$	$\bot$
ℓ	$\leftarrow$	$e \wedge \neg ab_e$	e	$\leftarrow$	$\top$
ℓ	$\leftarrow$	$o \wedge \neg ab_o$	e	$\leftarrow$	$\bot$
ab_e	$\leftarrow$	$\bot$	o	$\leftarrow$	$\top$
ab_o	$\leftarrow$	$\bot$	o	$\leftarrow$	$\bot$
ab_e	$\leftarrow$	$\neg o$			
ab_o	$\leftarrow$	$\neg e$			

Observations and Explanations

- An observation $\mathcal{O}$ is a set of ground literals
- $\mathcal{O}$ is explainable in the abductive framework $\langle \mathcal{P}, \mathcal{A}_{\mathcal{P}}, \mathcal{IC}, \models_{wcs} \rangle$
  iff there exists a non-empty $\mathcal{X} \subseteq \mathcal{A}_{\mathcal{P}}$ called explanation such that
  - $\mathcal{M}_{wc(\mathcal{P} \cup \mathcal{X})} \models_{wcs} L$ for all $L \in \mathcal{O}$
  - $\mathcal{M}_{wc(\mathcal{P} \cup \mathcal{X})}$ satisfies $\mathcal{IC}$
- Sometimes explanations are required to be minimal
  - where $\mathcal{X}$ is minimal if there does not exist an explanation $\mathcal{X}'$ with $\mathcal{X}' \subseteq \mathcal{X}$
- Is $\mathcal{P} \cup \mathcal{X}$ satisfiable?
- Is the empty observation explainable?

Observations and Explanations – Example

- Let $\mathcal{P}$ consist of

$$\begin{array}{rcl} happy & \leftarrow & cake \wedge \neg ab_{cake} \\ happy & \leftarrow & cookies \wedge \neg ab_{cookies} \\ ab_{cake} & \leftarrow & \bot \\ ab_{cookies} & \leftarrow & \bot \end{array}$$

- Then $\mathcal{A}_{\mathcal{P}}$ consists of

$$\begin{array}{rclcrcl} cake & \leftarrow & \top & \quad & cookies & \leftarrow & \top \\ cake & \leftarrow & \bot & \quad & cookies & \leftarrow & \bot \end{array}$$

- Let $\mathcal{IC} = \{\mathsf{U} \leftarrow cake \wedge cookies\}$
- Let $\mathcal{O} = \{happy\}$
- $\{cake \leftarrow \top\}$ and $\{cookies \leftarrow \top\}$ are explanations
- $\{cake \leftarrow \top, \; cookies \leftarrow \top\}$ is not an explanation

The Suppression Task – Experiments 7-9

Ex.	$\mathcal{P}$	$\mathcal{A}_\mathcal{P}$	$\mathcal{O}$	$\mathcal{X}$		e
7	$\ell \leftarrow e \wedge \neg ab_e$ $ab_e \leftarrow \bot$	$e \leftarrow \top$ $e \leftarrow \bot$	ℓ	$e \leftarrow \top$		0.71
8	$\ell \leftarrow e \wedge \neg ab_e$ $\ell \leftarrow t \wedge \neg ab_t$ $ab_e \leftarrow \bot$ $ab_t \leftarrow \bot$	$e \leftarrow \top$ $e \leftarrow \bot$ $t \leftarrow \top$ $t \leftarrow \bot$	ℓ	$e \leftarrow \top$	$t \leftarrow \top$	0.13
9	$\ell \leftarrow e \wedge \neg ab_e$ $\ell \leftarrow o \wedge \neg ab_o$ $ab_e \leftarrow \bot$ $ab_o \leftarrow \bot$ $ab_e \leftarrow \neg o$ $ab_o \leftarrow \neg e$	$e \leftarrow \top$ $e \leftarrow \bot$ $o \leftarrow \top$ $o \leftarrow \bot$	ℓ	$e \leftarrow \top$ $o \leftarrow \top$		0.54

The Suppression Task – Experiments 10-12

Ex.	$\mathcal{P}$	$\mathcal{A}_{\mathcal{P}}$	$\mathcal{O}$	$\mathcal{X}$		$\neg e$
10	$\ell \leftarrow e \wedge \neg ab_e$ $ab_e \leftarrow \bot$	$e \leftarrow \top$ $e \leftarrow \bot$	$\neg \ell$	$e \leftarrow \bot$		0.96
11	$\ell \leftarrow e \wedge \neg ab_e$ $\ell \leftarrow t \wedge \neg ab_t$ $ab_e \leftarrow \bot$ $ab_t \leftarrow \bot$	$e \leftarrow \top$ $e \leftarrow \bot$ $t \leftarrow \top$ $t \leftarrow \bot$	$\neg \ell$	$e \leftarrow \bot$ $t \leftarrow \bot$		0.96
12	$\ell \leftarrow e \wedge \neg ab_e$ $\ell \leftarrow o \wedge \neg ab_3$ $ab_e \leftarrow \bot$ $ab_3 \leftarrow \bot$ $ab_e \leftarrow \neg o$ $ab_3 \leftarrow \neg e$	$e \leftarrow \top$ $e \leftarrow \bot$ $o \leftarrow \top$ $o \leftarrow \bot$	$\neg \ell$	$e \leftarrow \bot$	$o \leftarrow \bot$	0.33

Skeptical and Credulous Consequences

- **Let $\langle \mathcal{P}, \mathcal{A}_{\mathcal{P}}, \mathcal{IC}, \models_{wcs} \rangle$ be an abductive framework, $\mathcal{O}$ an observation, and F a formula**
- ***F* follows credulously from $\mathcal{P}$ and $\mathcal{O}$**
  iff there exists an explanation $\mathcal{X}$ for $\mathcal{O}$ such that $\mathcal{P} \cup \mathcal{X} \models_{wcs} F$
- ***F* follows skeptically from $\mathcal{P}$ and $\mathcal{O}$**
  iff for all explanations $\mathcal{X}$ for $\mathcal{O}$ we find $\mathcal{P} \cup \mathcal{X} \models_{wcs} F$

Complementary Pairs

- A pair of clauses of the form $c \leftarrow \top$ and $c \leftarrow \bot$ is complementary
- A set of clauses is complementary if it contains a complementary pair
- Proposition 29 Let $\langle \mathcal{P}, \mathcal{A}_\mathcal{P}, \mathcal{IC}, \models_{wcs} \rangle$ be an abductive framework $\mathcal{O}$ an observation and $\mathcal{X} \subseteq \mathcal{A}_\mathcal{P}$ an explanation for $\mathcal{O}$ which contains a complementary pair $c \leftarrow \top$ and $c \leftarrow \bot$
  - Then, $\mathcal{X}' = \mathcal{X} \setminus \{c \leftarrow \bot\}$ is also an explanation for $\mathcal{O}$ and $\mathcal{M}_{wc(\mathcal{P} \cup \mathcal{X})} = \mathcal{M}_{wc(\mathcal{P} \cup \mathcal{X}')}$
- Proof $\rightsquigarrow$ Exercise
- Proposition 30 Given n undefined atoms in a ground program $\mathcal{P}$ there are 2^{2^n} subsets of $\mathcal{A}_\mathcal{P}$ and 3^n non-complementary subsets of $\mathcal{A}_\mathcal{P}$
- Proof $\rightsquigarrow$ Exercise
- Are humans considering $3^n - 1$ possible explanations?

Reasoning to the Best Explanation 1

- If I watered the garden, then the grass is wet
  If it was raining, then the grass is wet

- Reasoning towards a program

$$
\begin{array}{rcl}
wet\_grass & \leftarrow & watered \wedge \neg ab_{watered} \\
ab_{watered} & \leftarrow & \bot \\
wet\_grass & \leftarrow & rain \wedge \neg ab_{rain} \\
ab_{rain} & \leftarrow & \bot
\end{array}
$$

- Observation   The grass is wet

- What are the minimal explanations?

Reasoning to the Best Explanation 2

- If I watered the garden, then the grass is wet
  If it was raining, then the grass is wet
  The sky was clear all day
- Reasoning towards a program

$$\begin{array}{rcl} \mathit{wet\_grass} & \leftarrow & \mathit{watered} \wedge \neg \mathit{ab}_{\mathit{watered}} \\ \mathit{ab}_{\mathit{watered}} & \leftarrow & \bot \\ \mathit{wet\_grass} & \leftarrow & \mathit{rain} \wedge \neg \mathit{ab}_{\mathit{rain}} \\ \mathit{ab}_{\mathit{rain}} & \leftarrow & \bot \\ \mathit{clear\_sky} & \leftarrow & \top \end{array}$$

- Common sense $\mathsf{U} \leftarrow \mathit{clear\_sky} \wedge \mathit{rain}$
- Observation The grass is wet
- What is the best minimal explanation?

The Tweety Scenario 1

- Birds usually fly, but kiwis and penguins do not; Tweety and Jerry are birds
- Reasoning towards a program

$$\begin{array}{rcl} \textit{fly}\ X & \leftarrow & \textit{bird}\ X \wedge \neg \textit{ab}_{\textit{fly}}\ X \\ \textit{ab}_{\textit{fly}}\ X & \leftarrow & \textit{kiwi}\ X \\ \textit{ab}_{\textit{fly}}\ X & \leftarrow & \textit{penguin}\ X \\ \textit{bird tweety} & \leftarrow & \top \\ \textit{bird jerry} & \leftarrow & \top \end{array}$$

- The least model of its weak completion

$$\langle \{\textit{bird tweety},\ \textit{bird jerry}\}, \emptyset \rangle$$

- The set of abducibles

$$\begin{array}{rclcrcl} \textit{kiwi tweety} & \leftarrow & \top & \quad & \textit{kiwi tweety} & \leftarrow & \bot \\ \textit{kiwi jerry} & \leftarrow & \top & & \textit{kiwi jerry} & \leftarrow & \bot \\ \textit{penguin tweety} & \leftarrow & \top & & \textit{penguin tweety} & \leftarrow & \bot \\ \textit{penguin jerry} & \leftarrow & \top & & \textit{penguin jerry} & \leftarrow & \bot \end{array}$$

The Tweety Scenario 2

- **Birds usually fly, but kiwis and penguins do not; Tweedy and Jerry are birds**
- **Suppose we observe that *Jerry does fly***
- **The minimal explanation is**

$$\mathcal{X} \quad = \quad \{\textit{kiwi jerry} \leftarrow \bot, \ \textit{penguin jerry} \leftarrow \bot\},$$

- **The observation follows**
- **Are you happy with this formalization?**

The Tweety Scenario 3

- Birds usually fly; Tweety and Jerry are birds
- Reasoning towards a program

$$\begin{array}{rcl} \textit{fly } X & \leftarrow & \textit{bird } X \wedge \neg \textit{ab}_{\textit{fly}}\ X \\ \textit{ab}_{\textit{fly}}\ X & \leftarrow & \bot \\ \textit{bird tweety} & \leftarrow & \top \\ \textit{bird jerry} & \leftarrow & \top \end{array}$$

- The least model of its weak completion

$$\langle \{\textit{bird tweety},\ \textit{bird jerry},\ \textit{fly tweety},\ \textit{fly jerry}\}, \{\textit{ab}_{\textit{fly}}\ \textit{tweety},\ \textit{ab}_{\textit{fly}}\ \textit{jerry}\} \rangle.$$

- What is the set of abducibles in this case?
- Can the observation that Tweety does not fly be explained?
- Are you happy with this formalization?

Summary of Chapter 3

- **Programs as well as their weak completions admit least models under the three-valued Łukasiewicz logic**
  - **This does not hold if Kleene or Fitting logic is used**
- **The least models of weakly completed programs can be computed as least fixed points of an associated semantic operator**
- **These computations are bounded by the first limit ordinal in case of finite propositional programs, finite datalog programs or acyclic programs**
- **Abduction can be applied to explain observations**
  - **Humans seem to apply skeptical abduction**
- **The approach adequately models an average human reasoner in the suppression task**
- **All results hold in the presence of an equational theory**

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Human Reasoning and the Weak Completion Semantics

TECHNISCHE UNIVERSITÄT DRESDEN

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Applications and Extensions

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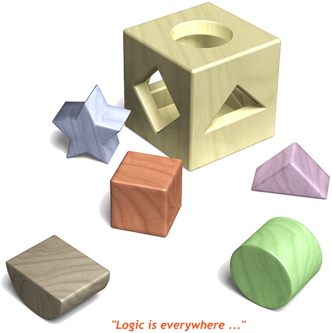
- **Conditional Reasoning**
- **Syllogistic Reasoning**
- **Disjunctive Reasoning**
- **Contextual Reasoning**
- **Spatial Reasoning**
- **Ethical Decision Problems**



"Logic is everywhere ..."

Conditional Reasoning

- **Conditionals**
- **The Semantics of Conditionals**
- **Reasoning with a Conditional**
- **Reasoning about a Conditional**
- **The Selection Task**

Introduction – Conditionals

- Conditionals are statements of the form *if antecedent then consequence*
- Claim of membership in a class or category
  - If it is a dog then it is a mammal
  - If the city is Rio then it is in Brasil
- Declarative (indicative) statements of fact or assumed fact
  - If the serial number is less that 150000 then it was built before 1995
  - If it is raining then the roofs are wet
  - If the roofs are wet then it is raining
- Promise
  - If you clean your shoes
    then Santa Claus will fill them with nuts, fruits, and chocolate
- Threat
  - If you violate the terms of the contract then we will sue

More Conditionals

- **Advice**
  - If it will be cold then put your sweater on
  - If it is raining then take your umbrella
- **Tip**
  - If you want to make a good impression then wear a dress or a suit and tie
- **Legal rules**
  - If you want to drink alcohol in a restaurant
    then you must be older than 18 years of age
- **Command**
  - If you find termites then apply the pesticide
- **Request**
  - If it is convenient for you
    then please drop the package off on your way to work

Even More Conditionals

- **Counterfactual**
  - If I had not taken this road today then I would have avoided the accident
- **Prediction**
  - If I take my umbrella then it will not rain in the afternoon
  - If there is a d on one side of a card then there is a 3 on the other side
- **Question**
  - If she graduates with 1
    will she be promoted to the PhD program of her choice?
- **Warning**
  - If you park there then your car will be towed
- Nickerson: Conditional Reasoning: 2015

Conditionals in this Lecture

- In the sequel, let *if $\mathcal{A}$ then $\mathcal{C}$* be a conditional, where
  - antecedent $\mathcal{A}$ and consequence $\mathcal{C}$
    are finite and consistent sets of ground literals
  - If $\mathcal{A}$ or $\mathcal{C}$ is a singleton set, then curly brackets are omitted
- Conditionals are evaluated wrt some background knowledge
  - a finite propositional or datalog program $\mathcal{P}$
  - an equational theory $\mathcal{E}$
  - a set of integrity constraints $\mathcal{IC}$
- Let $\mathcal{M}_{wc\mathcal{P}}$ be the least model of the weak completion of $\mathcal{P}$

The Semantics of Conditionals

- If it rains then the roofs are wet and she takes her umbrella
- Let $\mathcal{P}$ consist of

$$\begin{array}{rcl} wet\_roofs & \leftarrow & rain \wedge \neg ab_w \\ ab_w & \leftarrow & \bot \\ umbrella & \leftarrow & rain \wedge \neg ab_u \\ ab_u & \leftarrow & \bot \end{array}$$

- $\mathcal{M}_{wc\mathcal{P}} = \langle \emptyset, \{ab_w, ab_u\} \rangle \qquad \mathcal{A}_{\mathcal{P}} = \{rain \leftarrow \top,\ rain \leftarrow \bot\}$
- What follows if we additionally observe that
  - the roofs are wet?
  - she took her umbrella?
  - the roofs are not wet?
  - she did not take her umbrella?
- Are you happy with the formalization?

The Semantics of Conditionals – Obligation Conditionals

- A conditional *if* $\mathcal{A}$ *then* $\mathcal{C}$ is said to be an obligation conditional
  iff its consequence $\mathcal{C}$ is obligatory given its antecedent $\mathcal{A}$
- Byrne: The Rational Imagination: 2005
  - We cannot easily imagine a case
    where the antecedent is true and the consequence is not
  - The possibility $\mathcal{A} \wedge \neg\mathcal{C}$ is forbidden or unlikely
- Can you name obligation conditionals?
  - *If a person is drinking beer then the person must be over 19 years of age*
  - *If somebody is riding a motorbike then he/she must wear a helmet*
  - *If a german tourist wants to enter Russia then he needs a visa*
  - *If somebody's parents are elderly then he/she should look after them*
  - *If there is no light then plants will not grow*
  - *If an object is not supported it will drop to the floor*
  - *If it is raining then the roofs are wet*

Obligation Conditionals 2

- Byrne: The Rational Imagination: 2005
- **For obligation conditionals there are two initial possibilities people think about**
  - **the conjunction of antecedent and consequent (permitted)**
    - ***it rains and the roofs are wet***
  - **the conjunction of antecedent and negation of consequent (forbidden/unlikely)**
    - ***it rains and the roofs are not wet***
- **Exceptions are possible but unlikely**

Factual Conditionals

- A conditional *if* $\mathcal{A}$ *then* $\mathcal{C}$ is said to be a factual conditional iff its consequent $\mathcal{C}$ is not obligatory given its antecedent $\mathcal{A}$
- There is no forbidden or unlikely possibility
- Can you name factual conditionals?
  - *If the letter d is on one side of a card then the number 3 is on the other side*
  - *If Nancy rides her motorbike she goes to the mountains*
  - *If Fred was in Paris then Joe was in Lisbon*
  - *If it raining then she is taking her umbrella*
  - *If the sun is shining then I will water my garden in the evening*

Obligation versus Factual Conditionals – Summary

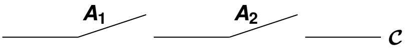
- Humans may classify conditionals as obligation or factual conditionals
- This is an informal and pragmatic classification
- It depends on
  - the background knowledge and experience of a human

  as well as on
  - the context in which a conditional is stated

Necessary Antecedents

- The antecedent $\mathcal{A}$ of a conditional *if* $\mathcal{A}$ *then* $\mathcal{C}$ is said to be necessary iff its consequent $\mathcal{C}$ cannot be true unless the antecedent is true
  - But the antecedent $\mathcal{A}$ may be true while the consequence $\mathcal{C}$ is not

series circuit



- Can you name conditionals with necessary antecedent?
  - *If the kid is tall enough then it can ride the roller coaster*
  - *If it is raining then the roofs are wet*
  - *If there is gas in the gas tank then the engine will start*
  - *If the switch is toggled then the light will be turned on*

Non-Necessary Antecedents

- **The antecedent $\mathcal{A}$ of a conditional *if* $\mathcal{A}$ *then* $\mathcal{C}$ is said to be non-necessary iff $\mathcal{A}$ is not necessary**
- **$\mathcal{C}$ may be true without $\mathcal{A}$ being true**
- **Can you name conditionals with non-necessary antecedent?**
  - ***If Polly is a parrot then Polly is a bird***
  - ***If the number ends with 3 then it is an odd number***
  - ***If the car has no gas then it will not run***
  - ***If it is raining then she is taking her umbrella***
  - ***If a person is drinking beer then the person must be over 19 years of age***
  - ***If the sun is shining then she is going to the swimming pool***
  - ***If I want to meet friends then I will go to my favorite pub***
  - ***If Nancy rides her motorbike she goes to the mountains***

Necessary versus Non-Necessary Antecedents – Summary

- Humans may classify antecedents as necessary or non-necessary
- The classification is informal and pragmatic
- It depends on
  - the background knowledge and experience of a human

  as well as on
  - the context in which a conditional is stated

Representing the Semantics of Conditionals

- Conditional *if A then C*
- Represented by

$$\begin{array}{lcl} C & \leftarrow & A \wedge \neg ab \\ ab & \leftarrow & \bot \end{array}$$

- Abducibles are

$$\mathcal{A}_\mathcal{P} = \{A \leftarrow \top,\ A \leftarrow \bot\}$$

- We extend the set of abducibles

$$\mathcal{A}^e_\mathcal{P} = \mathcal{A}_\mathcal{P} \cup \mathcal{A}^{nn}_\mathcal{P} \cup \mathcal{A}^f_\mathcal{P}$$

where

$$\begin{array}{lcll} \mathcal{A}^{nn}_\mathcal{P} & = & \{C \leftarrow \top \mid & C \text{ is head of a rule in } \mathcal{P} \text{ representing} \\ & & & \text{a conditional with non-necessary antecedent}\} \\ \mathcal{A}^{f}_\mathcal{P} & = & \{ab \leftarrow \top \mid & ab \text{ occurs in the body of a rule in } \mathcal{P} \\ & & & \text{representing a factual conditional}\} \end{array}$$

Returning to the Initial Example

$C \leftarrow A \wedge \neg ab$	A non-necessary	A necessary
Factual conditional	$ab \leftarrow \top,\ C \leftarrow \top$	$ab \leftarrow \top$
Obligation conditional	$C \leftarrow \top$	

- *If it rains then the roofs are wet*
  - **Obligation conditional with necessary antecedent**
  - $\mathcal{A}_\mathcal{P} = \{rain \leftarrow \top, rain \leftarrow \bot\} = \mathcal{A}^e_\mathcal{P}$
- *If it rains then she takes her umbrella*
  - **Factual conditional with non-necessary antecedent**
  - $\mathcal{A}^e_\mathcal{P} = \{rain \leftarrow \top,\ rain \leftarrow \bot,\ umbrella \leftarrow \top,\ ab_u \leftarrow \top\}$
- **Are you happier now?**

Reasoning with a Conditional

- **First premise: conditional sentence *if A then C***
- **Second premise: (possibly negated) atomic sentence**
  - **affirmation of the antecedent (AA)**
  - **denial of the antecedent (DA)**
  - **affirmation of the consequent (AC)**
  - **denial of the consequent (DC)**
- **What follows?**

Reasoning with a Conditional – Examples

- If it rains then the roofs must be wet
  It rains **(AA)**
- If Pauls rides a motorbike then Paul must wear a helmet
  Paul does not ride a motorbike **(DA)**
- If the library is open then Elisa is studying late in the library
  Elisa is studying late in the library **(AC)**
- If Nancy rides her motorbike then Nancy goes to the mountains
  Nancy does not go to the mountains **(DC)**
- **What follows?**

Facts, Assumptions, or Observations

- **First premise**

$$\begin{array}{rcl} C & \leftarrow & A \wedge \neg ab \\ ab & \leftarrow & \bot \end{array}$$

with set of abducibles

$$\mathcal{A} = \{A \leftarrow \top,\ A \leftarrow \bot\}$$

- **Shall the second premise be represented as fact, assumption, or observation?**
  - **So far, if atom undefined then fact or assumption else observation**
  - **In this section, always observation**

An Experiment

- 56 logically naive participants from mid-Europe including UK
- Proficient speakers in English
- They were given a short story and thereafter
  - a conditional sentence and a (possibly negated) atomic sentence
- What follows?
- 48 problems consisting of 12 conditionals classified by the authors
- Solved all four inference types (AA, DA, AC, DC)
- Participants could answer
  - corresponding atomic sentence which was not presented as second premise
  - corresponding negated atomic sentence
  - nothing (new) follows (nf)
- Participants acted as their own controls

Conditionals used in the Experiment

- **Obligation Conditionals with Necessary Antecedent**

  (1) **If it rains then the roofs must be wet**
  (2) **If water in the cooking pot is heated over 99° C then the water starts boiling**
  (3) **If the wind is strong enough then the sand is blowing over the dunes**

- **Obligation Conditionals with Non-Necessary Antecedent**

  (4) **If Paul rides a motorbike then Paul must wear a helmet**
  (5) **If Maria is drinking alcoholic beverages in a pub then Maria must be over 19 years of age**
  (6) **If it rains then the lawn must be wet**

- **Factual Conditionals with Necessary Antecedent**

  (7) **If the library is open then Sabrina is studying late in the library**
  (8) **If the plants get water then they will grow**
  (9) **If my car's start button is pushed then the engine will start running**

- **Factual Conditionals with Non-Necessary Antecedent**

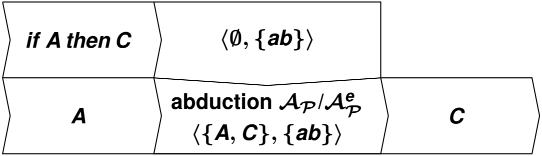
  (10) **If Nancy rides her motorbike then Nancy goes to the mountains**
  (11) **If Lisa plays on the beach then Lisa will get sunburned**
  (12) **If Ron scores a goal then Ron is happy**

Affirmation of the Antecedent (AA)

Class	*C*	¬*C*	*nf*	Sum	*Mdn C*	*Mdn nf*
(1)	**55**	**1**	**0**	**56**	**3343**	***na***
(2)	**55**	**1**	**0**	**56**	**3487**	***na***
(3)	**53**	**3**	**0**	**56**	**3516**	***na***
Obligation+necessary	**163 (.97)**	**5 (.03)**	**0**	**168**	**3408**	***na***
(4)	**53**	**1**	**2**	**56**	**3403**	**3472**
(5)	**53**	**2**	**1**	**56**	**3903**	**3572**
(6)	**54**	**1**	**1**	**56**	**3088**	**6959**
Obligation+non-necessary	**160 (.95)**	**4 (.02)**	**4 (.02)**	**168**	**3543**	**4183**
(7)	**49**	**1**	**6**	**56**	**3885**	**7051**
(8)	**54**	**1**	**1**	**56**	**3559**	**7349**
(9)	**54**	**1**	**1**	**56**	**3710**	**3826**
Factual+necessary	**157 (.93)**	**3 (.02)**	**8 (.05)**	**168**	**3615**	**6926**
(10)	**51**	**2**	**3**	**56**	**3929**	**6647**
(11)	**54**	**1**	**1**	**56**	**3777**	**5073**
(12)	**55**	**1**	**0**	**56**	**2977**	***na***
Factual+non-necessary	**160 (.95)**	**4 (.02)**	**4 (.02)**	**168**	**3644**	**5860**
Obligation	**323**	**9**	**4**	**336**	**3516**	**4183**
Factual	**317**	**7**	**12**	**336**	**3640**	**6575**
Necessary	**320**	**8**	**8**	**336**	**3546**	**6926**
Non-necessary	**320**	**8**	**8**	**336**	**3588**	**4934**
Total	**640 (.95)**	**16 (.02)**	**16 (.02)**	**672**	**3570**	**5925**

AA – Details

- $\mathcal{P} = \{C \leftarrow A \wedge \neg ab,\ ab \leftarrow \bot\}$ $\mathcal{A}_\mathcal{P} = \{A \leftarrow \top,\ A \leftarrow \bot\}$
- $\mathcal{O} = \{A\}$ is explained by $\{A \leftarrow \top\}$
- Neither $\{C \leftarrow \top\}$ nor $\{ab \leftarrow \top\}$ can explain $\mathcal{O}$



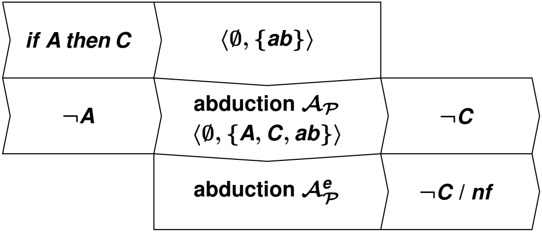
- **Please check an example for each class!**

Denial of the Antecedent (DA)

Class	*C*	¬*C*	*nf*	Sum	*Mdn* ¬*C*	*Mdn nf*
(1)	0	45	11	56	2863	4901
(2)	2	54	0	56	3367	*na*
(3)	2	51	3	56	3647	10477
Obligation+necessary	4 (0.2)	150 (.89)	14 (.08)	168	3356	5115
(4)	1	40	15	56	3722	7189
(5)	3	28	25	56	5735	7814
(6)	4	36	16	56	3602	6240
Obligation+non-necessary	8 (.05)	104 (.62)	56 (.33)	168	4064	7471
(7)	2	51	3	56	3928	7273
(8)	1	47	8	56	3296	5728
(9)	1	52	3	56	3549	8735
Factual+necessary	4 (.02)	150 (.89)	14 (.08)	168	3605	6582
(10)	1	39	16	56	3725	6874
(11)	0	41	15	56	3374	5887
(12)	1	41	14	56	3205	7002
Factual+non-necessary	2 (.01)	121 (.72)	45 (.28)	168	3374	6221
Obligation	12	254	70	336	3583	6613
Factual	6	271	59	336	3518	6221
Necessary	8 (.02)	300 (.89)	28 (.08)	336	3474	5808
Non-necessary	10 (.03)	225 (.67)	101 (.30)	336	3646	6700
Total	18 (.03)	525 (.78)	129 (.19)	672	3558	6450

DA – Details

- $\mathcal{P} = \{C \leftarrow A \wedge \neg ab,\ ab \leftarrow \bot\}$ $\mathcal{A}_{\mathcal{P}} = \{A \leftarrow \top,\ A \leftarrow \bot\}$
- $\mathcal{O} = \{\neg A\}$ is explained by
  - $\{A \leftarrow \bot\}$
  - $\{A \leftarrow \bot,\ C \leftarrow \top\}$ (in case of a non-necessary antecedent)



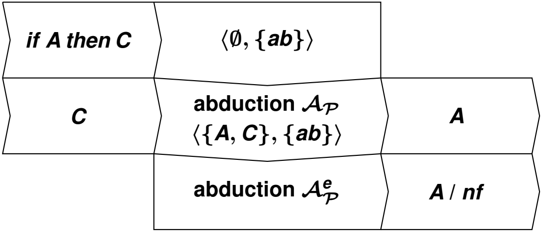
- **Please check an example for each class!**

Affirmation of the Consequent (AC)

Class	*A*	¬*A*	*nf*	Sum	*Mdn A*	*Mdn nf*
(1)	37	1	18	56	3952	7995
(2)	48	1	7	56	4003	4170
(3)	43	1	12	56	3458	9001
Obligation+necessary	128 (.76)	3 (.02)	37 (.22)	168	3797	8175
(4)	42	1	13	56	3659	8828
(5)	32	1	23	56	4704	6044
(6)	29	1	26	56	3593	4396
Obligation+non-necessary	103 (.61)	3 (.02)	62 (.37)	168	3968	5939
(7)	51	1	4	56	3767	4397
(8)	42	1	13	56	3798	4565
(9)	45	1	10	56	3492	4598
Factual+necessary	138 (.82)	3 (.02)	27 (.16)	168	3699	4565
(10)	34	2	20	56	5224	6289
(11)	29	2	25	56	3218	6205
(12)	33	1	22	56	3483	4992
Factual+non-necessary	96 (.57)	5 (.03)	67 (.40)	168	3885	6116
Obligation	231	6	99	336	3888	6044
Factual	234	8	94	336	3769	5650
Necessary	266 (.79)	6 (.02)	64 (.19)	336	3735	5450
Non-necessary	199 (.59)	8 (.02)	129 (.38)	336	3906	6039
Total	465 (.69)	14 (.02)	193 (.29)	672	3826	5802

AC – Details

- $\mathcal{P} = \{C \leftarrow A \wedge \neg ab,\ ab \leftarrow \bot\}$ $\mathcal{A}_\mathcal{P} = \{A \leftarrow \top,\ A \leftarrow \bot\}$
- $\mathcal{O} = \{C\}$ is explained by
  - $\{A \leftarrow \top\}$
  - $\{C \leftarrow \top\}$ (in case of a non-necessary antecedent)



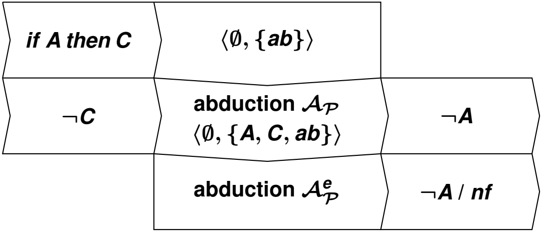
- **Please check an example for each class!**

Denial of the Consequent (DC)

Class	*A*	¬*A*	*nf*	Sum	*Mdn* ¬*A*	*Mdn nf*
(1)	**1**	**45**	**10**	**56**	**3449**	**4758**
(2)	**0**	**50**	**6**	**56**	**4058**	**7922**
(3)	**2**	**46**	**8**	**56**	**3796**	**4517**
Obligation+necessary	**3 (.02)**	**141 (.84)**	**24 (.14)**	**168**	**3767**	**5732**
(4)	**3**	**46**	**7**	**56**	**3872**	**4154**
(5)	**1**	**54**	**1**	**56**	**4946**	**8020**
(6)	**0**	**36**	**20**	**56**	**4062**	**5235**
Obligation+non-necessary	**4 (.02)**	**136 (.81)**	**28 (.17)**	**168**	**4293**	**5803**
(7)	**1**	**37**	**18**	**56**	**5974**	**4744**
(8)	**3**	**42**	**11**	**56**	**4367**	**5013**
(9)	**0**	**47**	**9**	**56**	**4208**	**3966**
Factual+necessary	**4 (0.2)**	**126 (.75)**	**38 (.23)**	**168**	**4849**	**4574**
(10)	**2**	**35**	**19**	**56**	**4879**	**4167**
(11)	**0**	**39**	**17**	**56**	**4411**	**5647**
(12)	**0**	**34**	**22**	**56**	**3726**	**3813**
Factual+non-necessary	**2 (.01)**	**108 (.64)**	**58 (.35)**	**168**	**4338**	**4542**
Obligation	**7 (.02)**	**277 (.82)**	**52 (.15)**	**336**	**4053**	**4790**
Factual	**6 (.02)**	**234 (.70)**	**96 (.28)**	**336**	**4459**	**4345**
Necessary	**7**	**267**	**62**	**336**	**4096**	**4758**
Non-necessary	**6**	**244**	**86**	**336**	**4325**	**4555**
Total	**13 (.02)**	**511(.76)**	**148 (.22)**	**672**	**4311**	**5162**

DC – Details

- $\mathcal{P} = \{C \leftarrow A \wedge \neg ab,\ ab \leftarrow \bot\}$ $\qquad \mathcal{A}_\mathcal{P} = \{A \leftarrow \top,\ A \leftarrow \bot\}$
- $\mathcal{O} = \{\neg C\}$ **is explained by**
  - $\{A \leftarrow \bot\}$
  - $\{ab \leftarrow \top\}$ **(in case of a factual conditional)**



- **Please check an example for each class!**

Reasoning About a Conditional

- Revision
- Minimal Revision Followed by Abduction
- Pam is Well
- The Moon is Not Made out of Cheese
- The Suppression Task Revisited
- The Shooting of Kennedy
- The Firing Squad
- The Forest Fire
- Relevance
- The Selection Task

Experiment – The Firing Squad

- Pearl: Causality: Models, Reasoning, and Inference: 2000
- *If the court orders an execution, then the captain will give the signal upon which riflemen A and B will shoot the prisoner*
  Consequently the prisoner will be dead
- **We assume that**
  - *the court's decision is unknown*
  - *both riflemen are accurate, alert, and law-abiding*
  - *the rifles are operating as expected*
  - *the prisoner is unlikely to die from any other causes*
- **Evaluate the following conditionals (true, false, unknown)**
  - *If the prisoner is not dead then the captain did not signal*
  - *If rifleman A shot then rifleman B shot as well*
  - *If rifleman A did not shoot then the prisoner is not dead*
  - *If the captain gave no signal and rifleman A decides to shoot, then the court did not order an execution*

More on Conditionals

- **In the sequel, let *if $\mathcal{A}$ then $\mathcal{C}$* be a conditional, where**
  - **antecedent $\mathcal{A}$ and consequence $\mathcal{C}$ are finite and consistent sets of ground literals**
- **Conditionals are evaluated wrt some background knowledge**
  - **a finite propositional or datalog program $\mathcal{P}$**
  - **an equational theory $\mathcal{E}$**
  - **a set of integrity constraints $\mathcal{IC}$ such that $\mathcal{M}_{wc\mathcal{P}}$ satisfies $\mathcal{IC}$**
- **We distinguish three cases wrt the value of the antecedent under $\mathcal{M}_{wc\mathcal{P}}$**

Indicative Conditionals

- Let *if* $\mathcal{A}$ *then* $\mathcal{C}$ be a conditional such that $\mathcal{M}_{wc\mathcal{P}}\ \mathcal{A} = \top$
  - Such conditionals are often called indicative conditionals
  - Their consequent is asserted to be true if their antecedent is true
  - Check whether $\mathcal{M}_{wc\mathcal{P}}\ \mathcal{C} = \top$ holds

Counterfactuals

- Let *if* $\mathcal{A}$ *then* $\mathcal{C}$ be a conditional such that $\mathcal{M}_{wc\mathcal{P}}\ \mathcal{A} = \bot$
  - Such conditionals are sometimes called counterfactuals
    - Their antecedent is false
    - Their consequent may or may not be true
    - But in the counterfactual circumstance of the antecedent being true the consequence is asserted to be true
  - *Counterfactuals are always true because the premise is false*
    Eco: The Name of the Rose: 1988
    - Humans do not consider counterfactuals this way
    - Counterfactuals are very important   Byrne: Counterfactuals in XAI: 2019
    - If the car had detected the pedestrian earlier and braked the passenger would not have been injured
    - If the car had not swerved and hit the wall the passenger would not have been injured
  - We need to revise the background knowledge

Revision

- Let $\mathcal{S}$ be a finite and consistent set of literals

$$rev(\mathcal{P}, \mathcal{S}) = (\mathcal{P} \setminus defs(\mathcal{P}, \mathcal{S})) \cup \mathcal{S}^{\uparrow}$$

  is called the revision of $\mathcal{P}$ with respect to $\mathcal{S}$

$$\begin{aligned} & rev(\{e \leftarrow \top,\ \ell \leftarrow e \wedge \neg ab_e,\ ab_e \leftarrow \bot\}, \{\neg \ell\}) \\ = \ & \{e \leftarrow \top,\ \ell \leftarrow \bot,\ ab_e \leftarrow \bot\} \end{aligned}$$

- **Proposition 31**
  Let $\mathcal{P}$ be a program, $\mathcal{E}$ an equational theory, and $\mathcal{S}$ a consistent set of literals
  - *rev* is nonmonotonic
  - If $\mathcal{M}_{wc\mathcal{P}}\, L = \mathsf{U}$ for all $L \in \mathcal{S}$ then *rev* is monotonic: $\mathcal{M}_{wc\mathcal{P}} \subseteq \mathcal{M}_{wc\ rev(\mathcal{P},\mathcal{S})}$
  - $\mathcal{M}_{wc\ rev(\mathcal{P},\mathcal{S})}\, \mathcal{S} = \top$
- Proof $\leadsto$ Exercise

Unknown Antecedents

- Let *if* $\mathcal{A}$ *then* $\mathcal{C}$ be a conditional such that $\mathcal{M}_{wc\mathcal{P}}\ \mathcal{A} = \mathsf{U}$
  - To the best of my knowledge this case has not been considered so far
  - We believe that humans would like to assign true to the antecedent
    - Skeptical abduction
    - Revision
  - There are scenarios where abduction alone cannot solve the problem
  - We propose to
    - minimally revise the background knowledge
    - and to apply skeptical abduction
    - such that the antecedent becomes true
  - Do humans make an attempt to assign false to the antecedent?

Minimal Revision Followed by Abduction (MRFA)

- Given $\mathcal{P}$, $\mathcal{E}$, $\mathcal{IC}$, and the conditional sentence *if* $\mathcal{A}$ *then* $\mathcal{C}$
- If $\mathcal{M}_{wc\mathcal{P}}$ does not satisfy $\mathcal{IC}$, then
  - if $\mathcal{O} = \emptyset$ can be explained by $\mathcal{X} \subseteq \mathcal{A}_{\mathcal{P}}$
    then evaluate *if* $\mathcal{A}$ *then* $\mathcal{C}$ with respect to $\mathcal{M}_{wc(\mathcal{P} \cup \mathcal{X})}$
    else *nothing follows*
- If $\mathcal{M}_{wc\mathcal{P}}\ \mathcal{A} = \top$, then the value of *if* $\mathcal{A}$ *then* $\mathcal{C}$ is $\mathcal{M}_{wc\mathcal{P}}\ \mathcal{C}$
- If $\mathcal{M}_{wc\mathcal{P}}\ \mathcal{A} = \bot$, then evaluate *if* $\mathcal{A}$ *then* $\mathcal{C}$ wrt $\mathcal{M}_{wc\ rev(\mathcal{P},\mathcal{S})}$, where
  - $\mathcal{S} = \{L \in \mathcal{A} \mid \mathcal{M}_{wc\mathcal{P}}\ L = \bot\}$
- If $\mathcal{M}_{wc\mathcal{P}}\ \mathcal{A} = \mathsf{U}$, then evaluate *if* $\mathcal{A}$ *then* $\mathcal{C}$ wrt $\mathcal{M}_{wc\mathcal{P}'}$, where
  - $\mathcal{P}' = rev(\mathcal{P}, \mathcal{S}) \cup \mathcal{X}$,
  - $\mathcal{S}$ is a minimal subset of $\mathcal{A}$,
  - $\mathcal{X} \subseteq \mathcal{A}_{rev(\mathcal{P},\mathcal{S})}$ is an explanation for $\mathcal{A} \setminus \mathcal{S}$
  - such that $\mathcal{P}' \models_{wcs} \mathcal{A}$ and $\mathcal{M}_{wc\mathcal{P}'}$ satisfies $\mathcal{IC}$
- Abduction has to be applied skeptically

Pam is well

- $\mathcal{P} = \{\textit{well} \leftarrow \top\}$
- $\mathcal{M}_{wc\mathcal{P}} = \langle\{\textit{well}\}, \emptyset\rangle$
- **Evaluate** ***if Pam is not well, then she has the flu***
- $\textit{rev}(\mathcal{P}, \neg\textit{well}) = \{\textit{well} \leftarrow \bot\}$
- $\mathcal{M}_{wc\ \textit{rev}(\mathcal{P}, \neg\textit{well})} = \langle\emptyset, \{\textit{well}\}\rangle$
- **Hence, the value of the conditional is unknown**
- **The conditional is not treated as an implication**

The Moon is Not Made out of Cheese

- $\mathcal{IC} = \{\bot \leftarrow \textit{cheese}\}$
- $\mathcal{P} = \emptyset$
- $\mathcal{M}_{wc\mathcal{P}} = \langle \emptyset, \emptyset \rangle$
- $\mathcal{X} = \{\textit{Cheese} \leftarrow \bot\}$ **explains** $\mathcal{O} = \emptyset$
- $\mathcal{M}_{wc(\mathcal{P} \cup \mathcal{X})} = \langle \emptyset, \{\textit{cheese}\} \rangle$
- **Evaluate** *if the moon is made out of cheese, then life exists on other planets*
- $\textit{rev}(\mathcal{P} \cup \mathcal{X}, \textit{cheese}) = \{\textit{cheese} \leftarrow \top\}$
- $\mathcal{A}_{\{\textit{cheese} \leftarrow \top\}} = \emptyset$
- *nothing follows*

The Suppression Task Revisited Again – Background Knowledge

- **In the remainder of this section** $\mathcal{E} = \mathcal{IC} = \emptyset$
- **Group 1**
  - ***If she has an essay to write then she will study late in the library***
- **Group 2**
  - ***If she has an essay to write then she will study late in the library***
  - ***If she has some textbooks to read then she will study late in the library***
- **Group 3**
  - ***If she has an essay to write then she will study late in the library***
  - ***If the library stays open then she will study late in the library***

The Suppression Task Revisited Again – Conditionals

- **The groups are asked to evaluate the following conditionals**
  - *If she has an essay to write then she will study late in the library*
    - $\mathcal{S} = \emptyset,\ \mathcal{X} = \{e \leftarrow \top\}$
  - *If she does not have an essay to write then she will not study late in the library*
    - $\mathcal{S} = \emptyset,\ \mathcal{X} = \{e \leftarrow \bot\}$
  - *If she will study late in the library then she has an essay to write*
    - Exercise
  - *If she will not study late in the library then she does not have an essay to write*
    - Exercise
- **Applying MRFA yields the same results as before**
  - Skeptical reasoning is required
  - It should be experimentally verified

The Shooting of Kennedy

- Adams: Subjunctive and indicative conditionals: 1970
- **Background knowledge**
  - *If Oswald shot then the president was killed*
  - *If somebody else shot then the president was killed*
  - *Oswald shot*
- **Reasoning towards a program $\mathcal{P}$**

$$\begin{array}{lllllllll} k & \leftarrow & os \wedge \neg ab_{os} & ab_{os} & \leftarrow & \bot & os & \leftarrow & \top \\ k & \leftarrow & ses \wedge \neg ab_{ses} & ab_{ses} & \leftarrow & \bot & & & \end{array}$$

- **Weakly completing $\mathcal{P}$ and computing $\mathcal{M}_{wc\mathcal{P}}$**

$$\langle \{os, k\}, \{ab_{os}, ab_{ses}\} \rangle$$

- **Evaluate**
  - *If Oswald did not shoot Kennedy in Dallas then no one else would have*
  - *If Kennedy was killed in Dallas and Oswald did not shoot then no one else would have*

The Shooting of Kennedy – The Set of Abducibles

- **Recall**

$$\begin{array}{lllllllll} k & \leftarrow & os \wedge \neg ab_{os} & ab_{os} & \leftarrow & \bot & os & \leftarrow & \top \\ k & \leftarrow & ses \wedge \neg ab_{ses} & ab_{ses} & \leftarrow & \bot & & & \end{array}$$

- **How would you classify the two conditionals of the background knowledge?**
  - **Factual conditionals with non-necessary antecedent**
- **Now consider**

  if Oswald shot or somebody else shot, then the president was killed

  - **Factual (generalized) conditional with necessary antecedent**
- **The set of abducibles**

$$\{ses \leftarrow \top,\ ses \leftarrow \bot,\ ab_{os} \leftarrow \top,\ ab_{ses} \leftarrow \top\}$$

  - $k \leftarrow \top$ **is not added**

The Shooting of Kennedy – First Conditional

- *If Oswald did not shoot Kennedy in Dallas then no one else would have*

  $$\textit{if } \neg os \textit{ then } \neg ses$$

- $rev(\mathcal{P}, \{\neg os\})$

  $$\begin{array}{lllllllll} k & \leftarrow & os \wedge \neg ab_{os} & ab_{os} & \leftarrow & \bot & os & \leftarrow & \bot \\ k & \leftarrow & ses \wedge \neg ab_{ses} & ab_{ses} & \leftarrow & \bot & & & \end{array}$$

- $\mathcal{M}_{wc\ rev(\mathcal{P},\{\neg os\})}$

  $$\langle \emptyset, \{os, ab_{os}, ab_{ses}\} \rangle$$

- **The counterfactual is unknown**

The Shooting of Kennedy – Second Conditional

- *If Kennedy was killed and Oswald did not shoot then no one else did*

$$if\ \{k, \neg os\}\ then\ \neg ses$$

- $rev(\mathcal{P}, \{\neg os\})$

$$\begin{array}{lllllllll} k & \leftarrow & os \wedge \neg ab_{os} & ab_{os} & \leftarrow & \bot & os & \leftarrow & \bot \\ k & \leftarrow & ses \wedge \neg ab_{ses} & ab_{ses} & \leftarrow & \bot & & & \end{array}$$

- $\mathcal{M}_{wc\ rev(\mathcal{P},\{\neg os\})}$

$$\langle \emptyset, \{os, ab_{os}, ab_{ses}\}\rangle$$

- $\mathcal{A}^{e}_{rev(\mathcal{P},\{\neg os\})}$

$$\{ses \leftarrow \top,\ ses \leftarrow \bot, ab_{os} \leftarrow \top,\ ab_{ses} \leftarrow \top\}$$

- $\mathcal{M}_{wc(rev(\mathcal{P},\{\neg os\}) \cup \{ses \leftarrow \top\})}$

$$\langle \{ses, k\}, \{os, ab_{os}, ab_{ses}\}\rangle$$

- **The counterfactual is false**

Modeling the Firing Squad

- **Reasoning towards a program $\mathcal{P}$**

$$\begin{array}{rcl} signal & \leftarrow & execution \wedge \neg ab_1 \\ rifleman_A & \leftarrow & signal \wedge \neg ab_2 \\ rifleman_B & \leftarrow & signal \wedge \neg ab_3 \\ dead & \leftarrow & rifleman_A \wedge \neg ab_4 \\ dead & \leftarrow & rifleman_B \wedge \neg ab_5 \\ alive & \leftarrow & \neg dead \wedge \neg ab_6 \end{array} \qquad \begin{array}{rcl} ab_1 & \leftarrow & \bot \\ ab_2 & \leftarrow & \bot \\ ab_3 & \leftarrow & \bot \\ ab_4 & \leftarrow & \bot \\ ab_5 & \leftarrow & \bot \\ ab_6 & \leftarrow & \bot \end{array}$$

- **Weakly completing the program and computing $\mathcal{M}_{wc\mathcal{P}}$**

$$\langle \emptyset, \{ab_1, ab_2, ab_3, ab_4, ab_5, ab_6\}\rangle$$

- **The set of abducibles $\mathcal{A}_{\mathcal{P}}$**

$$\{execution \leftarrow \top, \; execution \leftarrow \bot\}$$

  - $\mathcal{X}_\top = \{execution \leftarrow \top\}$ explains $\{signal, rifleman_A, rifleman_B, dead, \neg alive\}$
  - $\mathcal{X}_\bot = \{execution \leftarrow \bot\}$ explains $\{\neg signal, \neg rifleman_A, \neg rifleman_B, \neg dead, alive\}$
  - $\{\neg signal, rifleman_A\}$ cannot be explained

The Firing Squad – Conditionals

- **Recall**
  - $\mathcal{X}_\top = \{execution \leftarrow \top\}$ explains $\{signal, rifleman_A, rifleman_B, dead, \neg alive\}$
  - $\mathcal{X}_\bot = \{execution \leftarrow \bot\}$ explains $\{\neg signal, \neg rifleman_A, \neg rifleman_B, \neg dead, alive\}$
  - $\{\neg signal, rifleman_A\}$ cannot be explained
- *If the prisoner is alive then the captain did not signal*

$$\textit{if alive then} \neg signal : \mathcal{P} \mapsto \mathcal{P} \cup \mathcal{X}_\bot \mapsto \top$$

- *If rifleman A shot then rifleman B shot as well*

$$\textit{if } rifleman_A \textit{ then } rifleman_B : \mathcal{P} \mapsto \mathcal{P} \cup \mathcal{X}_\top \mapsto \top$$

- *If the captain gave no signal and rifleman A decides to shoot then the court did not order an execution*

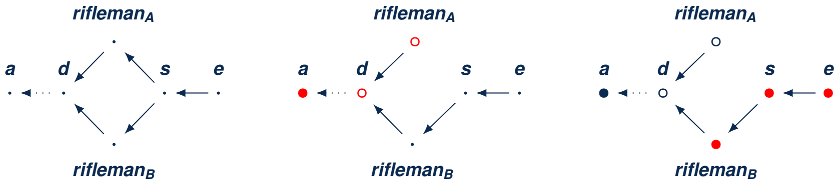
$$\textit{if } \{\neg signal, rifleman_A\} \textit{ then } \neg execution : \mathcal{P} \mapsto rev(\mathcal{P}, \{rifleman_A\}) \cup \mathcal{X}_\bot \mapsto \top$$

The Firing Squad – Last Conditional Revisited

- If the captain gave no signal and rifleman A decides to shoot then the court did not order an execution

$$\mathcal{P} \mapsto rev(\mathcal{P}, \{rifleman_A\}) \cup \mathcal{X}_\perp \mapsto \top$$

- Consider the dependency graphs (ignoring abnormalities)



· unknown ∘ true • false

The Forest Fire Example

- Byrne: The Rational Imagination: 2005
- **Suppose lightning hits a forest and a devastating forest fire breaks out**
  The forest was dry after a long hot summer and many acres were destroyed
- **Causal relationships** **lightning caused the forest fire**
- **Enabling relationships** **dry leaves made it possible for the fire to occur**
- **An enabler is usually not considered to be the cause for an event**
- **A missing enabler can prevent an event**

Encoding the Forest Fire Example

- *Lightning may cause a forest fire   Lightning happened   Dry leaves are present*

$$\mathcal{P} = \{ \; ff \leftarrow lightning \wedge \neg ab_\ell, \quad lightning \leftarrow \top, \quad ab_\ell \leftarrow \neg dryleaves, \quad dryleaves \leftarrow \top \}$$

- *If there had not been so many dry leaves on the forest floor then the forest fire would not have occurred*

	$\Phi_{\mathcal{P}}$	$\Phi_{rev(\mathcal{P},\{\neg dryleaves\})}$
$\uparrow 0$	$\langle \emptyset, \emptyset \rangle$	$\langle \emptyset, \emptyset \rangle$
$\uparrow 1$	$\langle \{dryleaves, lightning\}, \emptyset \rangle$	$\langle \{lightning\}, \{dryleaves\} \rangle$
$\uparrow 2$	$\langle \{dryleaves, lightning\}, \{ab_\ell\} \rangle$	$\langle \{lightning, ab_\ell\}, \{dryleaves\} \rangle$
$\uparrow 3$	$\langle \{dryleaves, lightning, ff\}, \{ab_\ell\} \rangle$	$\langle \{lightning, ab_\ell\}, \{dryleaves, ff\} \rangle$

$$rev(\mathcal{P}, \{\neg dryleaves\}) = \{ \; ff \leftarrow lightning \wedge \neg ab_\ell, \quad lightning \leftarrow \top, \quad ab_\ell \leftarrow \neg dryleaves, \quad dryleaves \leftarrow \bot \}$$

- **The counterfactual is true**

The Extended Forest Fire Example 1

- Pereira, Dietz, H.: Contextual Abductive Reasoning with Side-Effects: 2014
- **Add to the previous example** ***Arson may cause a forest fire***
- ***If there had not been so many dry leaves on the forest floor then the forest fire would not have occurred***

$$\begin{array}{rcl}
\mathcal{P} & = & \{ff \leftarrow lightning \wedge \neg ab_\ell,\ ff \leftarrow arson \wedge \neg ab_a, \\
 & & lightning \leftarrow \top,\ ab_\ell \leftarrow \neg dryleaves, \\
 & & dryleaves \leftarrow \top,\ ab_a \leftarrow \bot\} \\
\mathcal{M}_{wc\mathcal{P}} & = & \langle \{dryleaves, lightning, ff\}, \{ab_\ell, ab_a\} \rangle \\
rev(\mathcal{P}, \{\neg dryleaves\}) & = & \{ff \leftarrow lightning \wedge \neg ab_\ell,\ ff \leftarrow arson \wedge \neg ab_a, \\
 & & lightning \leftarrow \top,\ ab_\ell \leftarrow \neg dryleaves, \\
 & & dryleaves \leftarrow \bot,\ ab_a \leftarrow \bot\} \\
\mathcal{M}_{wc\, rev(\mathcal{P}, \{\neg dryleaves\})} & = & \langle \{lightning, ab_\ell\}, \{dryleaves, ab_a\} \rangle
\end{array}$$

- **The counterfactual is unknown**

The Extended Forest Fire Example 2

- ***If there had not been so many dry leaves on the forest floor***
  and there was no arson
  then the forest fire would not have occurred
- **What will happen?**

The Selection Task – The Abstract Case

- Wason: Reasoning About a Rule: 1968
- ***If the letter d is on one side of a card then the number 3 is on the other side***

d	**f**	**3**	**7**

- **Which cards must be turned to show that the rule holds?**
- **Humans typically turn the cards showing *d* and *3***

An Analysis of the Abstract Case

- Stenning, van Lambalgen: Human Reasoning and Cognitive Science: 2008
  - **With respect to classical two-valued logic!**
- **Almost everyone (89%) correctly selects *d***
  - **Corresponds to modus ponens in classical logic**
- **Almost everyone (84%) correctly does not select *f***
  - ***Because the condition does not mention f***
- **Many (62%) incorrectly select 3**
  - ***If there is a 3 on one side then there is a d on the other side***
  - **Converse of the given conditional**
- **Only a small percentage of participants (25%) correctly selects 7**
  - ***If the number on one side is not 3 then the letter on the other side is not d***
  - **Contrapositive of the given conditional**

The Selection Task – The Social Case

- Griggs, Cox: The Elusive Thematic Materials Effect in the Wason Selection Task: 1982
- ***If a person is drinking beer then the person must be over 19 years of age***

beer	coke	22yrs	16yrs

- **Which cards must be turned to show that the rule holds?**
- **Humans typically turn the cards showing *beer* and *16yrs***

The Selection Task – Alternative Conditional 1

- ***If Nancy rides her motorbike she goes to the mountains***

rides	no ride	mountain	no mountain

- **Which cards must be turned to show that the rule holds?**

The Selection Task – Alternative Conditional 2

- *If it rains then the roofs are wet*

rain	no rain	wet roofs	dry roofs

- **Which cards must be turned to show that the rule holds?**

The Selection Task

- **The Abstract Case**
  - ***If there is the letter d on one side of the card then the number 3 is on the other side***
    - **Factual conditional with necessary antecedent**
- **The Social Case**
  - ***If a person is drinking beer then the person must be over 19 years of age***
    - **Obligational conditional with non-necessary antecedent**

$C \leftarrow A \wedge \neg ab$	non-necessary	necessary
factual	$ab \leftarrow \top,\ C \leftarrow \top$	$ab \leftarrow \top$
obligational	$C \leftarrow \top$	

The Abstract Case: Factual Conditional with Necessary Antecedent

- *If the letter d is on one side of a card then there is the letter 3 on the other side*
- **Reasoning towards a program yields** $\mathcal{P} = \{3 \leftarrow d \wedge \neg ab_a,\ ab_a \leftarrow \bot\}$
- **Its set of abducibles is** $\mathcal{A}^e_{\mathcal{P}} = \{d \leftarrow \top,\ d \leftarrow \bot,\ ab_a \leftarrow \top\}$
- **Observations, least models, and decisions**

Observation d

true	false
d	ab_a
3	

turn
0.89

Observation $\neg d$

true	false
	d
	ab_a
	3

no turn
0.16

Observation 3

true	false
d	ab_a
3	

turn
0.62

Observation $\neg 3$

true	false
	d
	ab_a
	3

true	false
ab_a	
	3

no turn
0.25

The Social Case: Obligation with Non-Necessary Antecedent

- *If a person is drinking beer then the person must be over 19 years of age*
- **Reasoning towards a program yields** $\mathcal{P} = \{o \leftarrow b \wedge \neg ab_s,\ ab_s \leftarrow \bot\}$
- **Its set of abducibles is** $\mathcal{A}^e_\mathcal{P} = \{b \leftarrow \top,\ b \leftarrow \bot,\ o \leftarrow \top\}$
- **Observations, least models, and decisions**

Observation: b

true	false
b	ab_s
o	

turn
0.95

Observation: $\neg b$

true	false
	b
	ab_s
	o

no turn
0.025

Observation: o

true	false
o	ab_s

true	false
b	ab_s
o	

no turn
0.025

Observation: $\neg o$

true	false
	b
	ab_s
	o

turn
0.80

The Selection Task – Summary

- We obtain adequate answers if
  - the abstract case is interpreted as
    a factual conditional with necessary antecedent
  - the social case is interpreted as
    an obligational conditional with non-necessary antecedent
  - reasoning skeptically

Syllogisms

- Introduction
- A Meta-Study
- Seven Reasoning Principles
- The Representation of Quantified Statements
- Entailment
- Future Work

Introduction

- Consider the following inference

  In some cases when I go out, I am not in company
  Every time I am very happy I am in company
  Therefore, in some cases when I go out, I am not very happy

- It is valid
  - the conclusion is true in every case in which both premises are true
- Aristotle was the first to analyze syllogisms
- Syllogisms were central to logic until the second half of the 19th century
- Psychological studies of reasoning with determiners, such as *some* and *all*, have almost all concerned syllogistic reasoning

Reasoning

- **The ability to reason is at the core of human mentality**
- **Many contexts in daily life call for inferences**
  - **decisions about goals and actions**
  - **evaluation of conjectures and hypothesis**
  - **the pursuit of arguments and negotiations**
  - **the assessment of evidence and data**
  - **science, technology, and culture**
- **Examples**
  - ***Any experiment containing a confound is open to misinterpretation***
  - ***No current word processor spontaneously corrects a user's grammar***
  - ***Every chord containing three adjacent semitones is highly dissonant***

Common Sense Reasoning

- **In daily life, individuals reason in a variety of contexts, and often so rapidly that they are unaware of having made an inference**
- **Example**
  - ***Belinda: If you drop this cup it'll break***
  - ***Jeffrey: It looks pretty solid to me***
  - ***Belinda: Yes, but it's made from porcelain***

An Example

- **Try to determine, as quickly as you can, whether the following syllogism is valid**

  All roses are flowers
  Some flowers fade quickly
  Therefore, some roses fade quickly

- **Now, take your time and think about it again**

Another Example

- **What follows necessarily from the following premises?**

some a are b
no b are c

Aac	*all a are c*	
Iac	*some a are c*	
Eac	*no a are c*	**answer by humans**
Oac	*some a are not c*	**answer by humans the only correct answer wrt FOL**
Aca	*all c are a*	
Ica	*some c are a*	
Eca	*no c are a*	
Oca	*some c are not a*	
NVC	*no valid conclusion*	

Syllogisms

- **4 moods**

mood (*AFFIRMO NEGO*)	natural language	FOL	short
affirmative universal (A)	*all a are b*	$(\forall X)(a\,X \rightarrow b\,X)$	Aab
affirmative existential (I)	*some a are b*	$(\exists X)(a\,X \wedge b\,X)$	Iab
negative universal (E)	*no a are b*	$(\forall X)(a\,X \rightarrow \neg\, b\,X)$	Eab
negative existential (O)	*some a are not b*	$(\exists X)(a\,X \wedge \neg\, b\,X)$	Oab

- **4 figures**

	figure 1	figure 2	figure 3	figure 4
premise 1	a-b	b-a	a-b	b-a
premise 2	b-c	c-b	c-b	b-c

- **64 pairs of premises**
  - abbreviated by the first and the second mood and the figure (e.g., IE1)
- **512 syllogisms**
  - possible conclusions are the 4 moods instantiated by a-c and c-a

A Meta-Study

- Khemlani, Johnson-Laird 2012
- **Data from 6 studies**
  - **Humans deviate from FOL reasoning**
- **12 cognitive theories**
  - **None of the 12 theories models human reasoning adequately**
- ***The existence of 12 theories of any scientific domain is a small desaster***
- ***If psychologists could agree on an adequate theory of syllogistic reasoning, then progress towards a more general theory of reasoning would seem to be feasible***
- ***If researchers were unable to account for syllogistic reasoning, then they would have little hope of making sense of reasoning in general***

Three Examples

- **OA4:** ***some b are not a all b are c***
- **IE4:** ***some b are a no b are c***
- **IA2:** ***some b are a all c are b***

	participants	FOL	PSYCOP	mental models	verbal models
OA4	Oca	Oca	Oca	Oca	Oca
			Ica Iac	Oac NVC	NVC
matching percentage		1.0	0.78	0.78	0.89
IE4	Oac NVC	Oac	Oac	Oac NVC	Oac NVC
			Iac Ica	Eac Eca Oca	
matching percentage		0.89	0.67	0.67	1.00
IA2	Ica Iac			Ica Ica	Ica
		NVC	NVC	NVC	NVC
matching percentage		0.67	0.67	0.89	0.78
accuracy			0.77	0.83	0.84

Significance and Accuracy

- **Significance of an Answer**
  - Given 9 possible answers,
    the chance that a conclusion has been chosen randomly is 1/9 = 0.11
  - A binomial test shows that if a conclusion is drawn more than 0.16
    it is unlikely to be a random guess
- **Accuracy of the Predication**
  - For each syllogism
    - Order the nine possible conclusions (Aac, Eac, . . . , Oca, NVC)
    - Consider the list of the participant's conclusions (0, 1, . . . , 1, 0)
    - Compute the list of conclusions predicted by a theory (1, 0, . . . , 1, 1)
    - Compute
      $$comp\ i = \begin{cases} 1 & \text{if both lists have the same value for the } i\text{th element} \\ 0 & \text{otherwise} \end{cases}$$
    - The matching percentage of the syllogism is $\sum_{i=1}^{9} comp\ i/9$
  - The accuracy is the average of the matching percentage of all syllogisms

First Principle: Licenses for Inferences (licences)

- Stenning, van Lambalgen 2008
- **Formalize conditionals by licences for inferences**

$$\textbf{for all } X, \textbf{ if } q\,X \textbf{ then } p\,X$$

$$\Downarrow$$

$$p\,X \leftarrow q\,X \wedge \neg\, ab\,X$$
$$ab\,X \leftarrow \bot$$

Second Principle: Existential Import or Gricean Implicature (import)

- **Humans normally do not quantify over things that do not exist**
  - **Gricean implicature** Grice 1975
  - **Consequently, *for all* implies *there exists***
- **Likewise, humans seem to require existential import for a conditional to be true**
- **Furthermore, *some a are b* often implies *some a are not b***

Third Principle: Unknown Generalization (unknownGen)

- Humans seem to distinquish between *some a are b* and *all a are b*
- If we learn that *some a are b* then
  - there must be an object o_1 belonging to *a* and *b* (existential import)
  - there must be another object o_2 belonging to *a*
    and for which it is unknown whether it belongs to *b*
- This is a new principle!

Fourth Principle: Converse Interpretation (converse)

- **Some humans seem to distinguish between *some a are b* and *some b are a***
- **But in FOL** $\exists X(a\,X \wedge b\,X) \equiv \exists X(b\,X \wedge a\,X)$
- **Nevertheless, we propose that lab implies lba and vice versa**

Fifth Principle: Search Alternative Conclusions to NVC (abduction)

- Suppose, NVC is derived
  - Humans may not want to accept this conclusion
  - They proceed to check whether there exists unknown relevant information
  - This information may be explanations for facts
  - The facts will come from existential import
- Skeptical abduction

Sixth Principle: Negation by Transformation (transformation)

- Logic programs do not allow negative literals as heads of clauses
- Replace a negative conclusion $\neg\, p\, X$ by $p'\, X$ and add the clause

$$p\, X \leftarrow \neg\, p'\, X$$

as well as the weak integrity constraint

$$\mathsf{U} \leftarrow p\, X \wedge p'\, X$$

- Combined with the principle of licences for inferences we obtain

$$\begin{array}{rcl} p\, X & \leftarrow & \neg\, p'\, X \wedge \neg\, ab\, X \\ ab\, X & \leftarrow & \bot \\ \mathsf{U} & \leftarrow & p\, X \wedge p'\, X \end{array}$$

Seventh Principle: Blocking by Double Negatives (blocking)

- What conclusions can be drawn from double negatives?
- This appears to be a quite difficult reasoning task for humans
- They seem to avoid drawing conlusions through double negatives
- Example
  - *If not a then b*   *If not b then c*   *a is true*
  - We obtain

$$\begin{array}{rcl} b & \leftarrow & \neg a \wedge \neg ab_{nab} \\ ab_{nab} & \leftarrow & \bot \\ c & \leftarrow & \neg b \wedge \neg ab_{nbc} \\ ab_{nbc} & \leftarrow & \bot \\ a & \leftarrow & \top \end{array}$$

  - The least model of its weak completion is

$$\langle \{a, c\}, \{b, ab_{nab}, ab_{nbc}\} \rangle$$

  - c can be blocked by removing $ab_{nbc} \leftarrow \bot$

Ayz: All y are z

- $\mathcal{P}_{Ayz}$

$$\begin{array}{rcll} z\,X & \leftarrow & y\,X \wedge \neg\, ab_{yz}\,X & \text{licenses} \\ ab_{yz}\,X & \leftarrow & \bot & \text{licenses} \\ y\,o & \leftarrow & \top & \text{import} \end{array}$$

- **Computing the least model of its weak completion**

$$\begin{array}{rcll} \Phi_{\mathcal{P}_{Ayz}} \uparrow 0 & = & \langle \emptyset, \emptyset \rangle & \\ \Phi_{\mathcal{P}_{Ayz}} \uparrow 1 & = & \langle \{y\,o\}, \{ab_{yz}\,o\} \rangle & \\ \Phi_{\mathcal{P}_{Ayz}} \uparrow 2 & = & \langle \{y\,o, z\,o\}, \{ab_{yz}\,o\} \rangle & = \mathcal{M}_{wc\mathcal{P}_{Ayz}} \end{array}$$

Eyz: No y are z

- $\mathcal{P}_{Eyz}$

$$
\begin{array}{rcll}
z'\,X & \leftarrow & y\,X \wedge \neg\, ab_{ynz}\,X & \text{transformation\&licenses} \\
ab_{ynz}\,X & \leftarrow & \bot & \text{licenses} \\
y\,o & \leftarrow & \top & \text{import} \\
z\,X & \leftarrow & \neg\, z'\,X \wedge \neg\, ab_{nzz}\,X & \text{transformation\&licenses} \\
ab_{nzz}\,o & \leftarrow & \bot & \text{licenses\&blocking} \\
\\
\mathsf{U} & \leftarrow & z\,X \wedge z'\,X & \text{transformation}
\end{array}
$$

- **Computing the least model of its weak completion**

$$
\begin{array}{rcl}
\Phi_{\mathcal{P}_{Eyz}} \uparrow 0 & = & \langle \emptyset, \emptyset \rangle \\
\Phi_{\mathcal{P}_{Eyz}} \uparrow 1 & = & \langle \{y\,o\}, \{ab_{ynz}\,o, ab_{nzz}\,o\} \rangle \\
\Phi_{\mathcal{P}_{Eyz}} \uparrow 2 & = & \langle \{y\,o, z'\,o\}, \{ab_{ynz}\,o, ab_{nzz}\,o\} \rangle \\
\Phi_{\mathcal{P}_{Eyz}} \uparrow 3 & = & \langle \{y\,o, z'\,o\}, \{ab_{ynz}\,o, ab_{nzz}\,o, z\,o\} \rangle \quad = \quad \mathcal{M}_{wc\mathcal{P}_{Eyz}}
\end{array}
$$

Iyz: Some y are z

- $\mathcal{P}_{Iyz}$

$$
\begin{array}{rcll}
z\,X & \leftarrow & y\,X \wedge \neg\, ab_{yz}\,X & \text{licenses} \\
ab_{yz}\,o_1 & \leftarrow & \bot & \text{licenses\&unknownGen} \\
y\,o_1 & \leftarrow & \top & \text{import} \\
y\,o_2 & \leftarrow & \top & \text{unknownGen} \\
y\,X & \leftarrow & z\,X \wedge \neg\, ab_{zy}\,X & \text{converse\&licenses} \\
ab_{zy}\,o_3 & \leftarrow & \bot & \text{converse\&licenses\&unknownGen} \\
z\,o_3 & \leftarrow & \top & \text{converse\&import} \\
z\,o_4 & \leftarrow & \top & \text{converse\&unknownGen}
\end{array}
$$

- **Computing the least model of its weak completion**

$$
\begin{array}{rcl}
\Phi_{\mathcal{P}_{Iyz}} \uparrow 0 & = & \langle \emptyset, \emptyset \rangle \\
\Phi_{\mathcal{P}_{Iyz}} \uparrow 1 & = & \langle \{y\,o_1, y\,o_2, z\,o_3, z\,o_4\}, \{ab_{yz}\,o_1, ab_{zy}\,o_3\} \rangle \\
\Phi_{\mathcal{P}_{Iyz}} \uparrow 2 & = & \langle \{y\,o_1, y\,o_2, z\,o_3, z\,o_4, z\,o_1, y\,o_3\}, \{ab_{yz}\,o_1, ab_{zy}\,o_3\} \rangle \\
 & = & \mathcal{M}_{wc\mathcal{P}_{Iyz}}
\end{array}
$$

Oyz: Some y are not z

- $\mathcal{P}_{Oyz}$

$z'\,X$	$\leftarrow$	$y\,X \wedge \neg\, ab_{ynz}\,X$	transformation&licenses
$ab_{ynz}\,o_1$	$\leftarrow$	$\bot$	licenses&unknownGen
$y\,o_1$	$\leftarrow$	$\top$	import
$y\,o_2$	$\leftarrow$	$\top$	unknownGen
$z\,X$	$\leftarrow$	$\neg\, z'\,X \wedge \neg\, ab_{nzz}\,X$	transformation&licenses
$ab_{nzz}\,o_1$	$\leftarrow$	$\bot$	licenses&blocking
$ab_{nzz}\,o_2$	$\leftarrow$	$\bot$	licenses&blocking
U	$\leftarrow$	$z\,X \wedge z'\,X$	transformation

- **Computing the least model of its weak completion**

$$\Phi_{\mathcal{P}_{Oyz}} \uparrow 0 = \langle \emptyset, \emptyset \rangle$$

$$\Phi_{\mathcal{P}_{Oyz}} \uparrow 1 = \langle \{y\,o_1, y\,o_2\}, \{ab_{ynz}\,o_1, ab_{nzz}\,o_1, ab_{nzz}\,o_2\} \rangle$$

$$\Phi_{\mathcal{P}_{Oyz}} \uparrow 2 = \langle \{y\,o_1, y\,o_2, z'\,o_1\}, \{ab_{ynz}\,o_1, ab_{nzz}\,o_1, ab_{nzz}\,o_2\} \rangle$$

$$\Phi_{\mathcal{P}_{Oyz}} \uparrow 3 = \langle \{y\,o_1, y\,o_2, z'\,o_1\}, \{ab_{ynz}\,o_1, ab_{nzz}\,o_1, ab_{nzz}\,o_2, z\,o_1\} \rangle$$

$$= \mathcal{M}_{wc\mathcal{P}_{Oyz}}$$

Entailment of Syllogisms

- Khemlani, Johnson-Laird 2012 **appear to use entailment as defined in FOL**
- $\mathcal{P}$ **entails** Ayz **(all y are z)**

  iff $\exists X(\mathcal{P} \models_{wcs} y\,X) \wedge \forall X(\mathcal{P} \models_{wcs} y\,X \rightarrow \mathcal{P} \models_{wcs} z\,X)$
- $\mathcal{P}$ **entails** Eyz **(no y are z)**

  iff $\exists X(\mathcal{P} \models_{wcs} y\,X) \wedge \forall X(\mathcal{P} \models_{wcs} y\,X \rightarrow \mathcal{P} \models_{wcs} \neg\, z\,X)$
- $\mathcal{P}$ **entails** Iyz **(some y are z)**

  iff $\exists X_1(\mathcal{P} \models_{wcs} y\,X_1 \wedge z\,X_1) \wedge \exists X_2(\mathcal{P} \models_{wcs} y\,X_2 \wedge \mathcal{P} \not\models_{wcs} z\,X_2)$
  $\wedge\, \exists X_3(\mathcal{P} \models_{wcs} z\,X_3 \wedge \mathcal{P} \not\models_{wcs} y\,X_3)$
- $\mathcal{P}$ **entails** Oyz **(some y are not z)**

  iff $\exists X_1(\mathcal{P} \models_{wcs} y\,X_1 \wedge \neg z\,X_1) \wedge \exists X_2(\mathcal{P} \models_{wcs} y\,X_2 \wedge \mathcal{P} \not\models_{wcs} \neg z\,X_2)$
- $\mathcal{P}$ **entails NVC**

  iff **none of the above is entailed where either** $yz = ac$ **or** $yz = ca$

Syllogism OA4

- The premises are Oba (*some b are not a*) and Abc (*all b are c*)
- The participants concluded Oca (*some c are not a*)
- $\mathcal{P}_{OA4}$:

$b\ o_1$	$\leftarrow$	$\top$	import
$b\ o_2$	$\leftarrow$	$\top$	unknownGen
$a'\ X$	$\leftarrow$	$b\ X \wedge \neg\ ab_{bna}\ X$	transformation&licenses
$ab_{bna}\ o_1$	$\leftarrow$	$\bot$	unknownGen&licenses
$a\ X$	$\leftarrow$	$\neg a'\ X \wedge \neg\ ab_{naa}\ X$	transformation&licenses
$ab_{naa}\ o_1$	$\leftarrow$	$\bot$	blocking&licenses
$ab_{naa}\ o_2$	$\leftarrow$	$\bot$	blocking&licenses
$c\ X$	$\leftarrow$	$b\ X \wedge \neg\ ab_{bc}\ X$	licenses
$ab_{bc}\ X$	$\leftarrow$	$\bot$	licenses
$b\ o_3$	$\leftarrow$	$\top$	import
U	$\leftarrow$	$a\ X \wedge a'\ X$	transformation

- $\mathcal{M}_{wc\mathcal{P}_{OA4}} = \langle\{b\ o_1, b\ o_2, b\ o_3, a'\ o_1, c\ o_1, c\ o_2, c\ o_3\}, \{ab_{bna}\ o_1, ab_{naa}\ o_1, ab_{naa}\ o_2, ab_{bc}\ o_1, ab_{bc}\ o_2, ab_{bc}\ o_3, a\ o_1\}\rangle$
- $\mathcal{P}_{OA4}$ entails Oca and nothing else $\leadsto$ perfect match 1.0

Syllogism IE4

- The premises are Iba (*some b are a*) and Ebc (*no b are c*)
- The participants concluded Oac (*some a are not c*) and NVC
- $\mathcal{P}_{IE4}$:

$b\ o_1$	$\leftarrow$	$\top$	import
$b\ o_2$	$\leftarrow$	$\top$	unknownGen
$a\ X$	$\leftarrow$	$b\ X \wedge \neg\, ab_{ba}\ X$	licenses
$ab_{ba}\ o_1$	$\leftarrow$	$\bot$	licenses&unknownGen
$b\ X$	$\leftarrow$	$a\ X \wedge \neg\, ab_{ab}\ X$	converse&licenses
$ab_{ab}\ o_3$	$\leftarrow$	$\bot$	converse&licenses&unknownGen
$a\ o_3$	$\leftarrow$	$\top$	converse&import
$a\ o_4$	$\leftarrow$	$\top$	converse&unknownGen
$c'\ X$	$\leftarrow$	$b\ X \wedge \neg\, ab_{bnc}\ X$	transformation&licenses
$ab_{bnc}\ X$	$\leftarrow$	$\bot$	licenses
$c\ X$	$\leftarrow$	$\neg\, c'\ X \wedge \neg\, ab_{ncc}\ X$	transformation&licenses
$b\ o_5$	$\leftarrow$	$\top$	import
$ab_{ncc}\ X$	$\leftarrow$	$\bot$	licenses
U	$\leftarrow$	$c\ X \wedge c'\ X$	transformation

- $\mathcal{P}_{IE4}$ entails Oac and nothing else $\rightsquigarrow$ partial match 0.89

Syllogism IA2

- **The premises are Iba (*some b are a*) and Acb (*all c are b*)**
- **The participants concluded Iac and Ica**
- $\mathcal{P}_{IA2}$:

$a\,X$	$\leftarrow$	$b\,X \wedge \neg\, ab_{ba}\,X$	licenses
$ab_{ba}\,o_1$	$\leftarrow$	$\bot$	licenses&unknownGen
$b\,o_1$	$\leftarrow$	$\top$	import
$b\,o_2$	$\leftarrow$	$\top$	unknownGen
$b\,X$	$\leftarrow$	$a\,X \wedge \neg\, ab_{ab}\,X$	converse&licenses
$ab_{ab}\,o_3$	$\leftarrow$	$\bot$	converse&licenses&unknownGen
$a\,o_3$	$\leftarrow$	$\top$	converse&import
$a\,o_4$	$\leftarrow$	$\top$	converse&unknownGen
$b\,X$	$\leftarrow$	$c\,X \wedge \neg\, ab_{cb}\,X$	licenses
$ab_{cb}\,X$	$\leftarrow$	$\bot$	licenses
$c\,o_5$	$\leftarrow$	$\top$	import

- $\mathcal{M}_{wc\mathcal{P}_{IA2}} = \langle\{a\,o_1, a\,o_3, a\,o_4, b\,o_1, b\,o_2, b\,o_3, b\,o_5, c\,o_5\}$ $\{ab_{ba}\,o_1, ab_{ab}\,o_3, ab_{cb}\,o_1, ab_{cb}\,o_2, ab_{cb}\,o_3, ab_{cb}\,o_4, ab_{cb}\,o_5\}\rangle$
- $\mathcal{P}_{IA2}$ **entails NVC**
- **Search for alternatives** **skeptical abduction**

Syllogism IA2 Continued

- **Idea** the heads of existential imports are considered as observation

$$\mathcal{O} = \{b\,o_1,\ a\,o_3,\ c\,o_5\}$$

- The corresponding facts are removed

$$\mathcal{P}^-_{IA2} = \mathcal{P}_{IA2} \setminus \{b\,o_1 \leftarrow \top,\ a\,o_3 \leftarrow \top,\ c\,o_5 \leftarrow \top\}$$

- The minimal and skeptical explanation for $\mathcal{O}$ is

$$\mathcal{X} = \{c\,o_5 \leftarrow \top, c\,o_1 \leftarrow \top, c\,o_3 \leftarrow \top, ab_{ba}\,o_3 \leftarrow \bot\}$$

- Let $\mathcal{P}'_{IA2} = \mathcal{P}^-_{IA2} \cup \mathcal{X}$ and we obtain $\mathcal{M}_{wc\mathcal{P}'_{IA2}} =$

$$\langle\{a\,o_1, a\,o_3, a\,o_4, b\,o_1, b\,o_2, b\,o_3, b\,o_5, c\,o_1, c\,o_3, c\,o_5\}$$
$$\{ab_{ba}\,o_1, ab_{ba}\,o_3, ab_{ab}\,o_3, ab_{cb}\,o_1, ab_{cb}\,o_2, ab_{cb}\,o_3, ab_{cb}\,o_4, ab_{cb}\,o_5\}\rangle$$

- $\mathcal{P}'_{IA2}$ entails Iac and Ica and nothing else $\rightsquigarrow$ perfect match 1.0

The Examples Revisited

- **OA4:** ***some b are not a*** ***all b are c***
- **IE3:** ***some b are not a*** ***no b are c***
- **IA2:** ***some b are a*** ***all c are b***

	participants	FOL	PSYCOP	mental models	verbal models	WCS
OA4	**Oca**	**Oca**	**Oca**	**Oca**	**Oca**	**Oca**
			Ica Iac	**Oac NVC**	**NVC**	
		1.0	0.78	0.78	0.89	1.00
IE4	**Oac NVC**	**Oac**	**Oac**	**Oac NVC**	**Oac NVC**	**Oac**
			Iac Ica	**Eac Eca Oca**		
		0.89	0.67	0.67	1.00	0.89
IA2	**Ica Iac**			**Ica Iac**	**Ica**	**Ica Iac**
		NVC	**NVC**	**NVC**	**NVC**	
		0.67	0.67	0.89	0.78	1.00
accuracy			0.77	0.83	0.84	0.89

Discussion

- The best possible value achievable by WCS is .925
  - because NVC is entailed only if nothing else is entailed
- WCS is better than any other cognitive theory that I am aware of!
- Open Questions
  - How can we model clusters of reasoners?
  - How shall we define entailment?
  - What exactly is the role of the abnormalities?
  - How important is the sequence in which the premises are presented?
  - Is there a difference between abstract and social syllogisms?

Contextual Reasoning

- The Context Operator
- Contextual Programs
- Properties
- Examples

The Context Operator

- **A new truth-functional operator**

L	$ctxt\ L$
$\top$	$\top$
$\bot$	$\bot$
U	$\bot$

- **Captures locally negation by failure**

$$p \leftarrow q \qquad p \leftarrow \bot$$

$$p \leftarrow ctxt\ q \qquad p \leftarrow \bot$$

- **Their weak completions have the following minimal models**

$$\langle \emptyset, \emptyset \rangle \qquad \langle \emptyset, \{p\} \rangle$$

Another Example

- Let $\mathcal{P}_1 = \{p\,a \leftarrow \top,\ q\,b \leftarrow r\,b\}$ with $\mathcal{M}_{wc\mathcal{P}_1} = \langle\{p\,a\}, \emptyset\rangle$
  - How is $c\,\mathcal{P}_1$ defined?

$$c\,\mathcal{P}_1 = \{p\,a \leftrightarrow \top,\ p\,b \leftrightarrow \bot,\ q\,a \leftrightarrow \bot,\ q\,b \leftrightarrow r\,b,\ r\,a \leftrightarrow \bot,\ r\,b \leftrightarrow \bot\}$$

- Now consider $\mathcal{P}_2$

$$\begin{array}{rcl} p\,X & \leftarrow & X \approx a \\ q\,X & \leftarrow & X \approx b \wedge r\,b \\ X \approx X & \leftarrow & \top \end{array}$$

  - What is the least model of $wc\,\mathcal{P}_2$?

$$\mathcal{M}_{wc\mathcal{P}_2} = \langle\{a \approx a,\ b \approx b,\ p\,a\}, \emptyset\rangle$$

  - What happens if $\mathcal{P}_3 = \mathcal{P}_2 \cup \{a \approx b \leftarrow \bot,\ b \approx a \leftarrow \bot\}$?

$$\mathcal{M}_{wc\mathcal{P}_3} = \langle\{a \approx a,\ b \approx b,\ p\,a\}, \{a \approx b,\ b \approx a,\ p\,b,\ q\,a\}\rangle$$

  - Is there a problem with $\mathcal{P}_3$?

Another Example – Continued

- Let $\mathcal{P}_4$

$$\begin{array}{rcl} p\,X & \leftarrow & ctxt\,X \approx a \\ q\,X & \leftarrow & ctxt\,X \approx b \wedge r\,b \\ X \approx X & \leftarrow & \top \end{array}$$

  - **Can you specify a model of $wc\,\mathcal{P}_4$?**

$$\langle \{a \approx a,\ b \approx b,\ p\,a\}, \{p\,b,\ q\,a\}\rangle$$

  - **Compare**

$$\mathcal{M}_{wc\mathcal{P}_3} = \langle \{a \approx a,\ b \approx b,\ p\,a\}, \{a \approx b,\ b \approx a,\ p\,b,\ q\,a\}\rangle$$

  - **This is a local version of negation by failure!**

Contextual Programs

- **Literals are atoms or negated atoms**
- **Let *L* be a literal**
- **A contextual literal is of the form *ctxt L* or $\neg$ *ctxt L***
- **A contextual rule is of the form $A \leftarrow Body$, where *A* is an atom and *Body* is a finite conjunction of literals and contextual literals containing at least one contextual literal**
- **A contextual program is a set of rules, contextual rules, facts, and assumptions containing at least one contextual rule**
- **Note a program is not a contextual program**

Contextual Programs and Models

- $\mathcal{P}$

$$\begin{array}{lcl} p & \leftarrow & ctxt\ q \\ p & \leftarrow & \bot \end{array}$$

- $wc\mathcal{P}$

$$p \leftrightarrow ctxt\ q \vee \bot$$

- How many minimal models has $wc\mathcal{P}$?
- What is

$$\begin{array}{rcl} \langle \emptyset, \emptyset \rangle (wc\mathcal{P}) & = & ? \\ \langle \emptyset, \{p\} \rangle (wc\mathcal{P}) & = & ? \\ \langle \{p, q\}, \emptyset \rangle (wc\mathcal{P}) & = & ? \\ \langle \{p\}, \emptyset \rangle (wc\mathcal{P}) & = & ? \\ \langle \{q\}, \emptyset \rangle (wc\mathcal{P}) & = & ? \end{array}$$

- Does there exist a least model?

Contextual Programs and Supported Models

- Let $\mathcal{P}$ consist of

$$\begin{array}{lcl} p & \leftarrow & ctxt\ q \\ p & \leftarrow & \bot \end{array}$$

- $wc\mathcal{P}$ has two minimal models $\langle \emptyset, \{p\} \rangle$ and $\langle \{p, q\}, \emptyset \rangle$
- Let's apply the semantic operator

$\Phi_{\mathcal{P}}$	$I^\top$	$I^\bot$	$I^\top$	$I^\bot$
$\uparrow 0$			p q	
$\uparrow 1$		p	p	
$\uparrow 2$		p		p

- Only $\langle \emptyset, \{p\} \rangle$ is a fixed point
  - It will turn out that it is the only fixed point
  - It will be called supported model

Contextual Programs and Monotonicity

- Let $\mathcal{P}$ consist of

$$p \quad \leftarrow \quad ctxt\,\neg\,p$$

- We find

$\Phi_{\mathcal{P}}$	$I^{\top}$	$I^{\bot}$
$\uparrow 0$		
$\uparrow 1$		p
$\uparrow 2$	p	
$\uparrow 3$		p
$\vdots$	$\vdots$	$\vdots$

- The semantic operator is no longer monotonic
- $wc\mathcal{P} = \{p \leftrightarrow ctxt\,\neg p\}$ is unsatisfiable

Acyclic Contextual Programs

- Let L be a literal
$$lvl\ ctxt\ L = lvl\ \neg\ ctxt\ L = lvl\ L$$
- A contextual program $\mathcal{P}$ is acyclic with respect to the level mapping lvl if and only if for each rule $A \leftarrow Body$ occurring in $\mathcal{P}$ and each (normal or contextual) literal L occurring in $Body$ we find $lvl\ A > lvl\ L$
- A contextual program $\mathcal{P}$ is acyclic if and only if it is acyclic with respect to some level mapping
- Recall
$$d_{lvl}(I, J) = \begin{cases} \frac{1}{2^n} & I \neq J \text{ and} \\ & I\,A = J\,A \neq \text{U for all } A \text{ with } lvl\ A < n \text{ and} \\ & I\,A \neq J\,A \text{ or } I\,A = J\,A = \text{U for some } A \text{ with } lvl\ A = n \\ 0 & \text{otherwise} \end{cases}$$
- Proposition 25 still applies: d_{lvl} is a metric
- Proposition 26 still applies: $(\mathcal{I}, d_{lvl})$ is a complete metric space

Contextual Programs and Fixed Points 1

- In the sequel, let $\mathcal{P}$ be a contextual program, $\mathcal{E}$ and equational theory *lvl* a level mapping for $\mathcal{P}$ and $\mathcal{I}$ the set of interpretations for $\mathcal{P}$
- **Theorem 32** If $\mathcal{P}$ is acyclic with respect to *lvl* then $\Phi_{\mathcal{P}}$ is a contraction on the metric space $(\mathcal{I}, d_{lvl})$
- **Proof** Let I and J be interpretations, $\Phi = \Phi_{\mathcal{P}}$, and $d = d_{lvl}$
  - We will show $d(\Phi\, I, \Phi\, J) \leq \frac{1}{2} d(I, J)$
  - If $I = J$ then $\Phi\, I = \Phi\, J$ and $d(\Phi\, I, \Phi\, J) = d(I, J) = 0$
  - If $I \neq J$ then we find $n \in \mathbb{N}$ such that $d(I, J) \leq \frac{1}{2^n}$
    - We will show $d(\Phi\, I, \Phi\, J) \leq \frac{1}{2^{n+1}}$
    - i.e. for all ground atoms A with $lvl\, A < n + 1$ we find $\Phi(I)(A) = \Phi(J)(A) \neq \mathsf{U}$
    - Let's take some A with $lvl\, A < n + 1$
    - Because $\mathcal{P}$ is acyclic, for any $A \leftarrow L_1 \wedge \ldots \wedge L_m \in g\,\mathcal{P}$ we find $lvl\, L_i < lvl\, A < n + 1$ for all $1 \leq i \leq m$
    - Because $d(I, J) \leq \frac{1}{2^n}$ we find $I\, L_i = J\, L_i \neq \mathsf{U}$ for all $1 \leq i \leq m$
    - Hence, $\Phi(I)(A) = \Phi(J)(A) \neq \mathsf{U}$ □

Contextual Programs and Fixed Points 2

- **Proof of Theorem 27** If program $\mathcal{P}$ is acyclic with respect to *lvl* then $\Phi_{\mathcal{P}}$ is a contraction on the metric space $(\mathcal{I}, d_{lvl})$
  - can be proven as before
  - by considering non-contextual programs
- **Corollary 33** If $\mathcal{P}$ is acyclic then $\Phi_{\mathcal{P}}$ has a unique fixed point which can be computed by iterating $\Phi_{\mathcal{P}}$ up to ω times starting with any interpretation
  - Follows from Theorems 32 and 9 (Banach Contraction Mapping Theorem)

Contextual Programs and Fixed Points 3

- **Proposition 34**
  If $\mathcal{P}$ is acyclic then the unique fixed point of $\Phi_{\mathcal{P}}$ is a model of $wc\mathcal{P}$

- **Proof** Let $I = \langle I^{\top}, I^{\bot} \rangle$ be the unique fixed point of $\Phi = \Phi_{\mathcal{P}}$ and $A \leftrightarrow F \in wc\mathcal{P}$
  - $I\,A = \top$ We find $A \leftarrow Body \in g\mathcal{P}$ such that $I\,Body = \top$
    - Hence, $I\,F = I(A \leftrightarrow F) = \top$
  - $I\,A = \bot$ We find a clause $A \leftarrow Body \in g\mathcal{P}$ and
    for all clauses $A \leftarrow Body \in g\mathcal{P}$ we find $I\,Body = \bot$
    - Hence, $I\,F = \bot$ and $I(A \leftrightarrow F) = \top$
  - $I\,A = \mathsf{U}$ $\rightsquigarrow$ Exercise

- **Conjecture** the unique fixed point of $\Phi_{\mathcal{P}}$ a minimal model of $wc\mathcal{P}$

Supported Models

- The unique fixed point of $\Phi_{\mathcal{P}}$ is called supported model of $wc\mathcal{P}$
- It will be denoted by $\mathcal{M}_{wc\mathcal{P}}$
- Formula F follows from an acyclic contextual program $\mathcal{P}$ under WCS in symbols $\mathcal{P} \models_{wcs} F$ iff $\mathcal{M}_{wc\mathcal{P}}$ maps F to true
- Reconsider $\mathcal{P}$

$$\begin{array}{lll} p & \leftarrow & ctxt\ q \\ p & \leftarrow & \bot \end{array}$$

  - $\mathcal{M}_{wc\mathcal{P}} = \langle \emptyset, \{p\} \rangle$
  - $\mathcal{P} \models_{wcs} \neg p \wedge \neg(p \wedge q)$

The Tweety Scenario Revisited

- Let $\mathcal{P}$ consist of the following clauses:

$$
\begin{array}{rcl}
\mathit{fly}\ X & \leftarrow & \mathit{bird}\ X \wedge \neg ab_{\mathit{fly}}\ X \\
ab_{\mathit{fly}}\ X & \leftarrow & \mathit{ctxt}\ \mathit{kiwi}\ X \\
ab_{\mathit{fly}}\ X & \leftarrow & \mathit{ctxt}\ \mathit{penguin}\ X \\
\mathit{bird}\ \mathit{tweety} & \leftarrow & \top \\
\mathit{bird}\ \mathit{jerry} & \leftarrow & \top
\end{array}
$$

- Iterating the semantic operator yields

$\Phi_{\mathcal{P}}$	$I^{\top}$	$I^{\perp}$
$\uparrow 0$		
$\uparrow 1$	*bird tweety* *bird jerry*	ab_{fly} *tweety* ab_{fly} *jerry*
$\uparrow 2$	*bird tweety* *bird jerry* *fly tweety* *fly jerry*	ab_{fly} *tweety* ab_{fly} *jerry*

Tweety is a Penguin

- Suppose we learn that *Tweety is a penguin*
- Let $\mathcal{P}'$ be

$$
\begin{array}{rcl}
\textit{fly } X & \leftarrow & \textit{bird } X \wedge \neg \textit{ab}_{\textit{fly}}\ X \\
\textit{ab}_{\textit{fly}}\ X & \leftarrow & \textit{ctxt kiwi } X \\
\textit{ab}_{\textit{fly}}\ X & \leftarrow & \textit{ctxt penguin } X \\
\textit{bird tweety} & \leftarrow & \top \\
\textit{bird jerry} & \leftarrow & \top \\
\textit{penguin tweety} & \leftarrow & \top
\end{array}
$$

Computing the Supported Model

- Iterating the semantic operator yields

$\Phi_{\mathcal{P}'}$	$I^\top$	$I^\bot$
$\uparrow 0$		
$\uparrow 1$	*bird tweety* *bird jerry* *penguin tweety*	ab_{fly} *tweety* ab_{fly} *jerry*
$\uparrow 2$	*bird tweety* *bird jerry* *penguin tweety* ab_{fly} *tweety* *fly tweety* *fly jerry*	ab_{fly} *jerry*
$\uparrow 3$	*bird tweety* *bird jerry* *penguin tweety* ab_{fly} *tweety* *fly jerry*	ab_{fly} *jerry* *fly tweety*

The Drowning Problem

- **Drowning Problem** if an object belonging to a particular class and being exceptional with respect to some property of the class, becomes exceptional with respect to other or all properties of the class

- **Example**

$$
\begin{array}{rcl}
\textit{fly}\ X & \leftarrow & \textit{bird}\ X \wedge \neg\, \textit{ab}_{\textit{fly}}\ X \\
\textit{ab}_{\textit{fly}}\ X & \leftarrow & \textit{ctxt penguin}\ X \\
\textit{ab}_{\textit{fly}}\ X & \leftarrow & \textit{ctxt moa}\ X \\
\textit{wings}\ X & \leftarrow & \textit{bird}\ X \wedge \neg\, \textit{ab}_{\textit{wings}}\ X \\
\textit{ab}_{\textit{wings}}\ X & \leftarrow & \textit{ctxt moa}\ X \\
\textit{bird}\ t & \leftarrow & \top \\
\textit{penguin}\ t & \leftarrow & \top
\end{array}
$$

- **Least model of the weak completion**

$$\langle\{\textit{bird}\ t,\ \textit{penguin}\ t,\ \textit{ab}_{\textit{fly}}\ t,\ \textit{wings}\ t\}, \{\textit{fly}\ t,\ \textit{ab}_{\textit{wings}}\ t\}\rangle$$

- **The Weak Completion Semantics does not suffer from the drowning problem**

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Master programmes in Artificial Intelligence 4 Careers in Europe

Co-financed by the European Union
Connecting Europe Facility

This Master is run under the context of Action No 2020-EU-IA-0087, co-financed by the EU CEF Telecom under GA nr. INEA/CEF/ICT/A2020/2267423

Human Reasoning and the Weak Completion Semantics
Technische Universität Dresden
Exercise 8

Steffen Hölldobler, Meghna Bhadra

December 13, 2021

Note: Please consider the equational theory to be empty for each question, unless stated otherwise.

Problem 1

Within the WCS framework, what is the difference between $a \leftarrow \bot$ and $\bot \leftarrow a$?

Problem 2

Give an example of a program, and a set of integrity constraints such that neither the program itself, nor the weak completion of the program have models which satisfy the integrity constraints.

Problem 3

Please recall the definition of a *complementary* pair of clauses, and answer the following questions:

a. Prove the following proposition: Let $\langle \mathcal{P}, \mathcal{A}_{\mathcal{P}}, \mathcal{IC}, \models_{wcs} \rangle$ be an abductive framework, $\mathcal{O}$ an observation, and $\mathcal{X} \subseteq \mathcal{A}_{\mathcal{P}}$ an explanation for $\mathcal{O}$ which contains a complementary pair $c \leftarrow \top$ and $c \leftarrow \bot$. Then, $\mathcal{X}'{=}\mathcal{X} \setminus \{c \leftarrow \bot\}$ is also an explanation for $\mathcal{O}$ and $\mathcal{M}_{wcs(\mathcal{P}\cup\mathcal{X})}{=}\mathcal{M}_{wcs(\mathcal{P}\cup\mathcal{X}')}$.

b. What is the key takeaway?

Problem 4

Please consider the following scenario:

If Jill consumes a cold beverage then she feels good. If Jill consumes a hot beverage then she feels good. If Jill consumes chocolate then she feels good. Jill mostly avoids consuming a hot and a cold beverage in one meal. The observation here is that Jill feels good. The first three lines are represented by the following program,

$\{feelgood \leftarrow hotdrink \wedge \neg ab_{hot},\ ab_{hot} \leftarrow \bot,$
$feelgood \leftarrow colddrink \wedge \neg ab_{cold},\ ab_{cold} \leftarrow \bot,$
$feelgood \leftarrow chocolate \wedge \neg ab_{chocolate},\ ab_{chocolate} \leftarrow \bot\}.$

Given the concepts of strong and weak constraints from the lecture, please choose an appropriate one for this scenario and state the reason(s) for your choice.
Hint: Consider the minimal explanation(s) for the given observation.

Problem 5

Please consider the following program:

$\{fly(X) \leftarrow bird(X) \wedge \neg ab_{fly}(X),$
$ab_{fly}(X) \leftarrow kiwi(X),$
$ab_{fly}(X) \leftarrow penguin(X),$
$bird(tweety) \leftarrow \top,$
$bird(jerry) \leftarrow \top\}.$

a. What are the undefined (grounded) atoms?
b. What are the abducibles possible for the above undefined atoms?
c. What are the minimal explanations for an observation, Jerry cannot fly?

Problem 6

For each of the following, please write down a logic program $\mathcal{P}$, and an observation $\mathcal{O}$, and list one conclusion (formula) which follows:
a. Only credulously.
b. Only sceptically.
c. Both sceptically and credulously.
d. Neither sceptically nor credulously.

Human Reasoning and the Weak Completion Semantics
Technische Universität Dresden
Exercise 7

Steffen Hölldobler, Meghna Bhadra

December 7, 2021

Note: Please consider the equational theory to be empty for reach question, unless stated otherwise.

Problem 1

Consider the program P: $\{p \leftarrow \top,\ q \leftarrow \neg p\}$. The level mapping is, $level(p) = 0,\ level(q) = 1$.
Please answer the following questions:
a. Is the program acyclic? Why or why not?
b. Let $I_1 = \langle \emptyset,\ \emptyset \rangle$, $I_2 = \langle \{p\},\ \emptyset \rangle$ and $I_3 = \langle \{p\},\ \{q\} \rangle$.
Please compute $d_{level}(I_1,\ I_2)$ and $d_{level}(I_2,\ I_3)$.

Problem 2

Consider the program P: $\{p \leftarrow r \wedge q,\ q \leftarrow r \wedge p\}$. Please answer the following questions:
a. Is the program acyclic? Why or why not?
b. Is Φ_P a contraction? Why or why not?

Problem 3

Consider the program P: $\{even(0) \leftarrow \top,\ even(successor(X)) \leftarrow \neg even(X)\}$.
Let the level mapping be such that, $level(even(0)) = 0$,
$level(even(successor(0))) = 1,\ level(even(successor(successor(0)))) = 2$ and so on.
Please answer the following questions:
a. Is P acyclic? Why or why not?
b. Starting with the empty interpretation, please show some immediate consequences of Φ_P, namely, I_0 (this is the one *after* the empty interpretation), I_1, I_2, I_3.
c. What is the fixed point of P, I? Are any other fixed points possible?
c. Please compute the following, $d_{level}(I_0,\ I_1)$, $d_{level}(I_1,\ I_2)$, $d_{level}(I_2,\ I_3)$.
d. Please compute, $d_{level}(I_0,\ I_3)$ and $d_{level}(I_2,\ I_3)$.

Problem 4

a. Please state two points of difference between the operator Φ behaving as contraction and otherwise (not a contraction).
b. Based on whatever has been covered in the lectures so far, what is the utility of the Banach Contraction Mapping Theorem?

Human Reasoning and the Weak Completion Semantics
Technische Universität Dresden
Exercise 9

Steffen Hölldobler, Meghna Bhadra

January 3, 2022

Note: Please consider the equational theory and the set of integrity constraints to be empty for each question, unless stated otherwise.

Problem 1

Using relevant conditional sentences (preferably not from the manuscript) please differentiate between obligational and factual conditionals, and necessary and non-necessary antecedents.

Note: For the following questions, please illustrate all computational steps as specified in the current WCS framework.

Problem 2

Consider the (first) conditional premise: *if it is Christmas day, then I listen to Christmas carols*. Let us assume that it is classified as a factual conditional with non-necessary antecedent. Given the second premise,
a. *I do not listen to Christmas carols.*
b. *I listen to Christmas carols.*
What follows from each of the above sets of first and second premises?

- *It is Christmas day.*
- *It is not Christmas day.*
- *Nothing follows.*

Problem 3

Consider the conditional premise: *if the traffic signal shows a red light, then I stop my car at the signal*. Let us assume that it is classified as an obligational conditional with non-necessary antecedent. Given the second premise,
a. *I do not stop my car at the signal*. What follows?

- *The traffic signal shows a red light.*
- *The traffic signal does not show a red light.*
- *Nothing follows.*

b. *The traffic signal does not show a red light.* What follows?

- *I stop my car at the signal.*
- *I do not stop my car at the signal.*
- *Nothing follows.*

Problem 4

Consider the conditional premise: *if the lamp is switched on, then its bulb produces light.* Let us assume that it is classified as a factual conditional with necessary antecedent. Given the second premise,

a. *The lamp is switched on.* What follows?

- *Its bulb produces light.*
- *Its bulb does not produce light.*
- *Nothing follows.*

b. *The lamp's bulb does not produce light.* What follows?

- *The lamp is switched on.*
- *The lamp is not switched on.*
- *Nothing follows.*

c. *The lamp's bulb produces light.* What follows?

- *The lamp is switched on.*
- *The lamp is not switched on.*
- *Nothing follows.*

Human Reasoning and the Weak Completion Semantics II
Technische Universität Dresden
Exercise 6

Steffen Hölldobler, Meghna Bhadra

June 14, 2022

Problem 1

Please suggest a neural network where the role of modifiers are minimized or eliminated.

Problem 2

Can you think of a function that can be computed by a 3-layered feed forward network of logical threshold units but by a not 2-layered network of logical threshold units?

Human Reasoning and the Weak Completion Semantics II
Technische Universität Dresden
Exercise 7

Steffen Hölldobler, Meghna Bhadra

June 21, 2022

Problem 1

Given the program, $\mathcal{P}$:

$$\{e \leftarrow \top,\ o \leftarrow \bot,$$
$$l \leftarrow e \wedge \neg ab_e,$$
$$l \leftarrow o \wedge \neg ab_o,$$
$$ab_e \leftarrow \bot,\ ab_e \leftarrow \neg o,$$
$$ab_o \leftarrow \bot,\ ab_o \leftarrow \neg e\}.$$

Considering the active weighted connections from input-hidden and hidden-output layers to be w, and those from output-input layer to be 1, please construct a recurrent network, $\mathcal{N}_{\mathcal{P}}^{\circlearrowleft}$ and:

- Illustrate every time step of the computation of its unique stable state while highlighting the activation patterns in each,
- Mention the sum of the weighted inputs of the active units in the output layer during the appropriate time steps.

Human Reasoning and the Weak Completion Semantics II
Technische Universität Dresden
Exercise 8

Steffen Hölldobler, Meghna Bhadra

June 28, 2022

Problem 1

Given the program, $\mathcal{P}$:

$$\{happy \leftarrow tea \wedge \neg ab_t, ab_t \leftarrow \bot,$$

$$happy \leftarrow coffee \wedge \neg ab_c, ab_c \leftarrow \bot,$$

$$cookies \leftarrow \neg cake,$$

$$milk \leftarrow cookies,$$

$$tea \leftarrow \top\},$$

and the integrity constraint, $\{\mathsf{U} \leftarrow tea \wedge coffee\}$.
Please answer the following questions by constructing appropriate networks:

1. Determine or detect whether $\mathcal{P}$ has reached a stable state.
2. Check whether the integrity constraint is satisfied.
3. Check if a given observation, $\mathcal{O} = \{\neg cake\}$, can be explained by the least model of $wc(\mathcal{P})$.
4. Provide an externally activated clamp unit to extend the network, such that $\mathcal{O}$ can be (minimally) explained.
5. Does the above lead to any stable coalition? Give reasons for your response.

Human Reasoning and the Weak Completion Semantics II
Technische Universität Dresden
Exercise 9

Steffen Hölldobler, Meghna Bhadra

July 5, 2022

Problem 1

Given the program, $\mathcal{P}$:

$$\{relax \leftarrow tea \wedge \neg ab_t, ab_t \leftarrow \bot\}$$

and the empty integrity constraint.
Please answer the following questions by constructing appropriate networks:

1. Provide a finite automaton generating all possible and non-complementary explanations. Do state how the output function encodes the explanations.
2. Why are we interested in non-complementary explanations?
3. What are *sceptical* conclusions?
4. Provide a McCulloch-Pitts network for the finite automaton.
5. Consider the observation, $\mathcal{O} = \{relax\}$. Extend the previous network (you can omit the recurring network details where the stable state is generated) such that the new network generates *sceptical* conclusions for $\mathcal{O}$. Highlight all the active units in the input and output layers of the last network.
6. Under what conditions will the various input units of the last network be activated?
7. Why is each unit that you have highlighted in the output layer of the last network, active?
8. Imagine there are multiple explanations for $\mathcal{O}$. How or why would the corresponding multiple least models persist in the input layer of the last network?

Human Reasoning and the Weak Completion Semantics II
Technische Universität Dresden
Exercise 1

Steffen Hölldobler, Meghna Bhadra

January 25, 2022

Note: Please consider the equational theory to be empty for each question, unless stated otherwise.

Problem 1

1. Please create a contextual program $\mathcal{P}$ representing the following scenario and compute the least fixed point:
   Birds usually fly. However, Penguins and Kiwis are birds which do not. Birds usually have wings, but Kiwis do not. Tweety is a Penguin. Sylvester is a bird.
2. Is the operator $\Phi_{\mathcal{P}}$ monotonic?
3. Is $\mathcal{P}$ acyclic?
4. Is the above computed model the least model of $wc(\mathcal{P})$, or is it a supported one?

Problem 2

Discuss whether Proposition 35 holds for non-contextual acyclic programs as well.

Human Reasoning and the Weak Completion Semantics II
Technische Universität Dresden
Exercise 2

Steffen Hölldobler, Meghna Bhadra

May 4, 2022

Note: Please consider the equational theory and the set of integrity constraints to be empty for each question, until stated otherwise.

Problem 1

In this exercise we will try to simulate the core idea behind Schrödinger's famous thought experiment. Hence, please consider the following Wikipedia excerpt on the same: *"To further illustrate, Schrödinger described how one could, in principle, create a superposition in a large-scale system by making it dependent on a quantum particle that was in a superposition. He proposed a scenario with a cat in a locked steel chamber, wherein the cat's life or death depended on the state of a radioactive atom, whether it had decayed and emitted radiation or not. According to Schrödinger, the Copenhagen interpretation implies that the cat remains both alive and dead until the state has been observed. Schrödinger did not wish to promote the idea of dead-and-live cats as a serious possibility; on the contrary, he intended the example to illustrate the absurdity of the existing view of quantum mechanics."*.

The following statements (somewhat) depicts the scenario:
If there is no observer looking inside the steel chamber then the cat is dead. If there is no observer looking inside the steel chamber then the cat is alive. If there is a quantum collapse of the radioactive atom then it releases some poison. If any poison is released then the cat is dead. If any poison is not released then the cat is alive.

- Let the set of integrity constraints be empty. What would be the least models of (minimal) abduction applied to the empty observation?
- Now considering the *realistic* constraint that a cat can either be dead or alive but not both, which models persist?
- Given the above constraint, and the fact that Schrödinger's experiment showed the absurdity of basing the superposition of the cat's living state on the presence (or not) of an observer, which statements do you think may be removed from the prior set of statements?
- Would you have formulated the simulation differently? If so, please elaborate.

Human Reasoning and the Weak Completion Semantics II
Technische Universität Dresden
Exercise 4

Steffen Hölldobler, Meghna Bhadra

May 17, 2022

Problem 1

Let us consider a modified version of the *trolley problem.* The modifications are the following:

- There are two humans on the side track and one on the main track.
- There is something about the side track that slows down the speed of the trolley in such a way that the trolley will *injure* the humans, instead of killing them. This implies that on changing the switch there will be injured people instead of dead people.

Suppose that in such a case the person controlling the (track) switch, Hank, prefers the direct action *change* over *donothing*. Please propose and discuss how you would model such a preference. In particular please do not leave out any clause from the program being proposed.
(Good to remember - *casualty* means people badly affected by an event or situation, here, dead or injured people).

Human Reasoning and the Weak Completion Semantics II
Technische Universität Dresden
Exercise 5

Steffen Hölldobler, Meghna Bhadra

May 24, 2022

Problem 1

- Please discuss and provide a modelling of the *collapsing bridge* scenario for utilitarianism and the doctrines of double and triple effects.
- As an extension to the above, assume that something is *abnormal* with respect to the collapse, for example the switch of the machine which will collapse the bridge is not working. Because of this the actor in the scenario, Ian, now considers if he should throw the heavy person off the bridge. To that end, please provide a modelling of the extension under the doctrine of the double effect.

(Good to remember - *casualty* means people badly affected by an event or situation, here, dead or injured people).

Human Reasoning and the Weak Completion Semantics II
Technische Universität Dresden
Exercise 3

Steffen Hölldobler, Meghna Bhadra

May 9, 2022

Problem 1

Consider a docking terminal for ships which has three blocks, a, b and c placed on a dock, and a robot arm which can help one move these blocks around.
Given an initial state of the world: $\mathit{ondock}(a) \circ \mathit{ondock}(b) \circ \mathit{on}(c,a) \circ \mathit{clear}(b) \circ \mathit{clear}(c) \circ \mathit{empty}$.
The fluent $\mathit{ondock}(X)$ signifies that block X is on the dock, $\mathit{on}(X, Y)$ signifies that block X is on top of Y, $\mathit{clear}(X)$ signifies that the top of block X is clear, empty signifies that the robot's arm is not holding anything and $\mathit{holding}(X)$ signifies that the robot is holding X.
We want to reach the goal state: $\mathit{ondock}(c) \circ \mathit{on}(b,c) \circ \mathit{on}(a,b) \circ \mathit{clear}(a) \circ \mathit{empty}$.

Given the following set of possible actions, $\mathit{action}(\mathit{preconditions}, \mathit{name}, \mathit{effects})$:
$\mathit{action}(\mathit{clear}(V) \circ \mathit{ondock}(V) \circ \mathit{empty}, \mathit{pickup}(V), \mathit{holding}(V))$,
$\mathit{action}(\mathit{clear}(V) \circ \mathit{on}(V, W) \circ \mathit{empty}, \mathit{unstack}(V, W), \mathit{holding}(V) \circ \mathit{clear}(W))$,
$\mathit{action}(\mathit{holding}(V), \mathit{putdown}(V), \mathit{clear}(V) \circ \mathit{ondock}(V) \circ \mathit{empty})$,
$\mathit{action}(\mathit{holding}(V) \circ \mathit{clear}(W), \mathit{stack}(V, W), \mathit{on}(V, W) \circ \mathit{clear}(V) \circ \mathit{empty})$.

Also given the set of rules:
$\mathit{causes}(X, [\,], Y) \leftarrow X \approx (Y \circ Z)$,
$\mathit{causes}(X, [V|W], Y) \leftarrow \mathit{action}(P, V, Q) \wedge (P \circ Z) \approx X \wedge \mathit{causes}(Z \circ Q, W, Y)$,
$X \approx X$.

Please answer the following questions:

- Using the above set of actions and rules of inference, starting with

$$\mathit{causes}(\mathit{ondock}(a) \circ \mathit{ondock}(b) \circ \mathit{on}(c,a) \circ \mathit{clear}(b) \circ \mathit{clear}(c) \circ \mathit{empty}, W,$$
$$\mathit{ondock}(c) \circ \mathit{on}(b,c) \circ \mathit{on}(a,b) \circ \mathit{clear}(a) \circ \mathit{empty}),$$

  show the next 3 steps of computation. *Recall: $\mathit{causes}(X, W, Y)$ signifies the sequence of actions W which transforms state X to state Y.*

- List the plan or the (grounded) sequence of actions (not to be confused with the *action* predicate) which would transform the given initial state to the desired goal state.

- Suppose that the symbol $\circ$ is idempotent, meaning $X \circ X \approx X$. Using one of the above steps of computation as an example show what would change and what kind of problem could arise.

Human Reasoning and the Weak Completion Semantics
Technische Universität Dresden

Steffen Hölldobler, Meghna Bhadra

November 9, 2021

Problem 1

Normal logic programs are finite or countably infinite set of clauses of the form Head if Body i.e. $Head \leftarrow A_1 \wedge A_2 \wedge \ldots \wedge A_m \wedge \neg B_1 \wedge \neg B_2 \wedge \ldots \wedge \neg B_n$*,* $Head \leftarrow \top$ *or* $Head \leftarrow \bot$ *where* $Head$ *is a positive literal (also called atom). The right hand side of the implication* $\leftarrow$ *is called the* $Body$*.*
Now, consider the normal logic program P_1: $\{e \leftarrow \top,\ l \leftarrow e \wedge \neg ab_e,\ ab_e \leftarrow \bot\}$. *Clauses are usually taken to be a finite collection of (universally closed) positive or negative literals (i.e. atoms or their negations respectively) which are connected using disjunctions, i.e.* $l_1 \vee l_2 \vee \ldots \vee l_n$*.*
Each of the implications in the program P_1 are logically equivalent to clauses.
a. Can you write down the (equivalent) clausal form for each?
b. *A definite logic program is a special kind of logic program that does not have occurrences of negations or* $\bot$. Then, is P_1 a definite logic program? Why or why not?
c. P_1 is a propositional program. What will be a grounded instance of P_1?

Problem 2

Consider the definite first-order program P_2: $\{p(a) \leftarrow \top,\ p(f(X)) \leftarrow p(X)\}$. X is a variable, a is a constant symbol and f is a function symbol. We only consider two-valued logic.
a. What is the Herbrand Universe? Is it finite?
b. What is the Herbrand Base?
c. What is the grounded program?
d. A Herbrand interpretation maps atoms in the Herbrand Base to true or false. It is this mapping that distinguishes one Herbrand interpretation from another. An Herbrand interpretation is called a (Herbrand) model when it maps each clause in the grounded program to true. In two-valued logic, models can be represented simply using the atoms from the Base which the model has mapped to true. Furthermore, in two-valued logic, the model intersection property holds for definite logic programs. This basically means, the intersection of all Herbrand models of a definite logic program in two-valued logic is again a model. This model is called the least model, as it is minimal and there can be no further minimal models aside from this.
Given all this information, what should be the least (Herbrand) model for the above program?
e. Recall the definition of the dependency function *deps*. What is $deps(P_2,\ p(a))$ and $deps(P_2,\ \neg p(f(f(a))))$?

Problem 3

a. Proposition: *Given a definite logic program P. The model intersection property holds for P, in two-valued logic; in other words P will have a least Herbrand model.* Prove the proposition.

b. Consider what would happen to the model intersection property in case of a normal logic program as defined above, w.r.t. two-valued logic.

Problem 4

Consider the normal first-order logic program P_3: $\{q(X) \leftarrow \neg p(X),\ p(a) \leftarrow \top\}$ and equation $a \approx b$.
a. What will be the Herbrand Universe, and the Herbrand Base?
b. Under the semantics of the three-value Łukasiewicz-logic, what might be the least model of P_3?
c. Can you also think of a model which is not the least one?

Problem 5 (Optional)

Let us revisit the Supression Task. We see how weak completion (under the three valued semantics of Łukasiewicz-logic) helps us adequately model the responses of the experiment. But what if the weak completions are replaced by completions? Do you think it could still sufficiently model the Supression Task?

Human Reasoning and the Weak Completion Semantics
Technische Universität Dresden
Exercise 5

Steffen Hölldobler, Meghna Bhadra

November 22, 2021

Problem 1

a. Give an example of a partially ordered set where the *least upper bound* does not exist.
b. Give an example of a partially ordered set where the *greatest lower bound* does not exist.

Problem 2

With reference to the notions of *monotonic* and *continuous* functions as discussed in the lecture and the manuscript, give an example of a partially ordered set S (please also mention the partial order itself)*, and specify a function $f : S \to S$ if possible, which is:
a. Monotonic.
b. Non-Monotonic.
c. Continuous.
d. Not Continuous.
f. Monotonic and Continuous.
e. Continuous and Non-Monotonic.
***Note**: You are free to use different partially ordered sets for each question, if you like.

Problem 3

Consider the set $S = \{0, 1, 2, 3, \ldots, \omega, \omega + 1, \omega + 2, \ldots, \omega + \omega\}$, and the relation $\leq$. Here, the symbol ω is the *first limit ordinal* that occurs after the set of natural numbers, followed by the non-limit ordinals $\omega + 1, \omega + 2$ etc. The symbol $\omega + \omega$ is the *second limit ordinal.* Consider a function $f : S \to S$, such that $f(x) = x$ if $x < \omega$ and $f(x) = \omega + \omega$ if $x \geq \omega$. Now, please answer the following questions:
a. Is the set S partially ordered?
b. Is the function f monotonic? Why or why not?
c. Is the function f continuous? Why or why not?

Human Reasoning and the Weak Completion Semantics
Technische Universität Dresden
Exercise 6

Steffen Hölldobler, Meghna Bhadra

November 29, 2021

Problem 1

Consider the program P: $\{q(X) \leftarrow \neg p(X),\ p(a) \leftarrow \top\}$, and the equational theory $\{a \approx b\}$.
a. What is the Herbrand Base, and the grounded program?
b. Starting from the empty interpretation $\langle \emptyset,\ \emptyset \rangle$, please show how the least fixed point of this program can be computed by the (iterative) application of the modified Fitting operator, Φ_P.
c. What do you think is the least model of P?
d. What do you think is the least model of the weak completion of P?
e. What observation can be drawn from the above points b, c and d?
f. With particular regard to the above point b, is Φ_P *monotonic*? Please state why.
g. With particular regard to the above point b, is Φ_P *continuous*? (Hint: You can use propositions from the manuscript to justify your response.)
h. What is the set of all *possible interpretations* (not to be confused with *models*) of P?
i. Is the above set (let's call it I) directed? Please state the reason(s) for your answer.
j. Is I a complete partial order? Please state the reason(s) for your answer.
k. Please write down any two directed subsets of I, and state their least upper bounds.

Problem 2

a. Consider the program P_1: $\{q(1) \leftarrow \top,\ q(X \circ a) \leftarrow q(X)\}$ and the AC1 theory: $\{x \circ 1 \approx X,\ X \circ Y \approx Y \circ X,\ (X \circ Y) \circ Z \approx X \circ (Y \circ Z)\}$. Please state the Herbrand Universe, the Herbrand Base, the grounded program, and state the least fixed point by listing the first 5 iterations of the Φ_{P_1} operator.
b. Consider the program P_2: $\{q(1) \leftarrow \top,\ q(X \circ a) \leftarrow q(X),\ p \leftarrow \neg q(X)\}$, and the above AC1 theory. Please state the Herbrand Universe, the Herbrand Base, the grounded program, and state the least fixed point by listing the first 5 iterations of the Φ_{P_2} operator.
c. Is there a difference in the number of iterations between the two operators, before it reaches a fixed point?

Problem 3

Consider the program $\{q(a) \leftarrow \top,\ r(b) \leftarrow \top, p(X) \leftarrow q(X) \wedge r(X)\}$, and the equation $\{b \approx c\}$. Please state the following,
a. Herbrand Universe.
b. Herbrand Base.
c. Grounded program.
d. A bijection between elements from the Herbrand Base, to propositional atoms (of your

choice).
e. The resulting, equivalent propositional program, P.
f. Starting with the empty interpretation, please show the computation of the least fixed point of the propositional program, using Φ_P.
g. Is Φ_P monotonic, and continuous? Please state the reasons for your response.

Problem 4

Please provide a proof sketch of the following proposition: *Let X be a directed (sub)set of interpretations. Then, the interpretation $I = \langle \bigcup X^\top, \bigcup X^\perp \rangle$ is the least upper bound of X.* Note: Here, $\bigcup X^\top$ and $\bigcup X^\perp$ denote the union of all the true and false elements of all interpretations in the (sub)set X, respectively.

Human Reasoning and the Weak Completion Semantics
Technische Universität Dresden
Exercise 4

Steffen Hölldobler, Meghna Bhadra

November 16, 2021

Problem 1

Consider the formula P: $a \leftrightarrow b$.
a. What will be the least three-valued model under the semantics of Łukasiewicz-logic?
b. What will be the other non-least three-valued models under the same semantics?
c. Is a, $\neg a$, b or $\neg b$ a *logical consequence* of P? Please state the reasons for your response.

Problem 2

In the Suppression Task as presented by the WCS manuscript, the conditional used by Byrne in the original experiment, namely *if she has an essay to write, then she must study in the library* seems a bit outdated. This is because, nowadays going to the library is not always so important or mandatory when one has an essay to write. One can simply resort to the internet, for example. Therefore, in view of broader research questions and applications of the Weak Completion Semantics, how the Suppression Task would work in a more modern context and setting is something we would like to consider. The findings may help supplement further research and development of the framework.
To that end, the goal of this exercise is to consider the purpose of the Suppression Task conducted by Ruth Byrne (you can refer to the WCS manuscript) and then imagine it in a modern context. In other words please think of a context (possibly different from the original), such that when replacing the conditionals with those from the context that your propose, we could possibly imitate the original experiment. Also, please consider what the responses should be for the AA, DA, AC and DC tasks, w.r.t the new context and conditionals.

Human Reasoning and the Weak Completion Semantics
Technische Universität Dresden
Exercise 10

Steffen Hölldobler, Meghna Bhadra

January 12, 2022

Note: Please consider the equational theory and the set of integrity constraints to be empty for each question, unless stated otherwise.

Problem 1

Given the following: *If X is a bird then X usually flies. However, Kiwis and Penguins cannot fly. Tweety is a bird.*
This can be represented by the program P: $fly(X) \leftarrow bird(X) \wedge \neg ab_{fly}(X)$, $ab_{fly} \leftarrow kiwi(X)$, $ab_{fly} \leftarrow penguin(X)$, $bird(X) \leftarrow kiwi(X)$, $bird(X) \leftarrow penguin(X)$, $bird(Tweety) \leftarrow \top$.
Also given: *Tweety can fly.*
You are provided the three choices of responses:

- *Tweety is either a kiwi or a penguin.*
- *Tweety is neither a kiwi nor a penguin.*
- *Nothing follows.*

Assuming the classification of the conditional *if X is a bird then X usually flies* to be factual conditional with non-necessary antecedent, what do you think would be the general and sceptical responses of humans? Can you model both?

Problem 2

Given the following: *If X is a bird then X usually flies. Jonathan is a bird.*
This can be represented by the program P: $fly(X) \leftarrow bird(X) \wedge \neg ab_{fly}(X)$, $ab_{fly} \leftarrow \bot$, $bird(Jonathan) \leftarrow \top$.
Assume the conditional is classified as factual with non-necessary antecedent. Given the second premise: *Jonathan does not fly.* Please model an explanation for the said observation.

Problem 3

Consider the experiments 7 and 8 of the suppression task. How would *you* classify the conditionals? Please remodel the said experiments taking your classification into account. For each, also state if the new conclusion differs from the one in the original experiment.

Problem 4

a. Consider experiment 2 of the suppression task. We slightly change the background knowledge to: *if she has an essay to write then she will study late in the library, if she has textbooks to read then she will study late in the library.* Assume that the conditionals are classified as obligational with non-necessary antecedent. Please show the evaluation of the conditional, *if she has an essay to write then she will study late in the library* using MRFA. Is the given conditional *indicative* or *subjunctive*?

b. Consider experiment 9 of the suppression task. Assume that the conditional *if she has an essay to write then she will study late in the library* has been classified as obligational with non-necessary antecedent. And the conditional *if the library is open then she will study late in the library* has been classified as factual with necessary antecedent. Please show the evaluation of the conditional, *if she is studying late in the library then she has an essay to write* using MRFA. Is the given conditional *indicative* or *subjunctive*?

Human Reasoning and the Weak Completion Semantics
Technische Universität Dresden
Exercise 11

Steffen Hölldobler, Meghna Bhadra

January 18, 2022

Note: Please consider the equational theory and the set of integrity constraints to be empty for each question, unless stated otherwise.

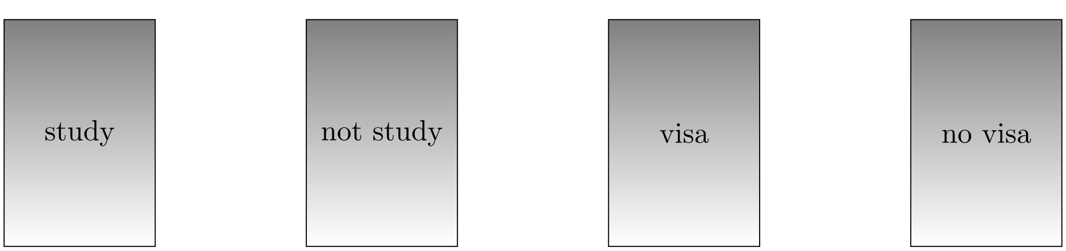
Problem 1

Given the four cards below, you are provided with the conditional: *if there is an A on one side of the card, then there is a 9 on the other side.*

Which two cards would *you* turn in order to check whether the given conditional is indeed true? Please show the modeling process explicitly and explain your choices.
(*Your classification for this particular conditional need not match with how it is in the manuscript*).

Problem 2

Given the four cards below, you are provided with the conditional: *if a person with an Indian citizenship wants to study in Germany, then the person needs a visa.* This is classified as an obligation with non-necessary antecedent.



Which two cards would you turn in order to check whether the given conditional is indeed true? Please show the modeling process explicitly and explain your choices.

Problem 3

Given the following background knowledge: *if it rains, then the roofs are wet and she takes her umbrella*. The first part of the information is an obligation but not the second.
Please show the modeling of the evaluation of the following conditionals (starting with the specification of the program) along with the possible dependency graphs:
a. *If the roofs are not wet then it has not rained.*
b. *If she has taken her umbrella then it has rained.*
(*The said dependency graphs may also be sent as clear photos or scans alongside the solutions. Please do not forget to mention the numbering in such a case*).

Human Reasoning and the Weak Completion Semantics
Technische Universität Dresden
Exercise 12

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January 25, 2022

Note: Please consider the equational theory to be empty for each question, unless stated otherwise.

Problem 1

Illustrate how the following syllogisms can be modeled within the WCS framework. Please also state which principle is used for which clause in the corresponding programs and finally what the syllogism entails (as currently defined within the framework):

- AO1,
- EI2,
- OE4.